1 Assignments: May 19th, 2026

Exercise 1.1.
Prove: Let G be a bipartite graph with a partition (A, B) such that $|A| = |B| = n$ and

$$e(G) = \frac{n^2}{2} - 1.$$

The graph G contains a subgraph $K_{\frac{n}{2},\frac{n}{2}}$ if and only if (A, B) is not a $\frac{1}{2}$-regular pair.

Proof. We assume that n is even.

$$d(X, Y) = \frac{e(X, Y)}{|X||Y|}.$$

The density of the whole pair is

$$d(A, B) = \frac{e(G)}{n^2} = \frac{1}{2} - \frac{1}{n^2}.$$

First, we suppose that G contains a copy of $K_{\frac{n}{2},\frac{n}{2}}$. Let its two vertex classes be $X \subseteq A$ and $Y \subseteq B$, where $|X| = |Y| = \frac{n}{2}$. Then $d(X, Y) = 1$, and hence

$$|d(X, Y) - d(A, B)| = 1 - \left(\frac{1}{2} - \frac{1}{n^2}\right) = \frac{1}{2} + \frac{1}{n^2} > \frac{1}{2}.$$

Thus (A, B) is not $\frac{1}{2}$-regular.

Conversely, suppose that (A, B) is not $\frac{1}{2}$-regular. Then there exist $X \subseteq A$ and $Y \subseteq B$ with

$$|X| \geq \frac{n}{2}, \qquad |Y| \geq \frac{n}{2}$$

such that

$$|d(X, Y) - d(A, B)| \geq \frac{1}{2}.$$

Since $d(A, B) < \frac{1}{2}$ and $d(X, Y) \geq 0$, the deviation cannot occur below $d(A, B)$. Therefore

$$d(X, Y) - d(A, B) \geq \frac{1}{2},$$

and so

$$d(X, Y) \geq 1 - \frac{1}{n^2}.$$

We claim that (X, Y) must be complete. If one edge between X and Y were missing, then

$$d(X, Y) \leq 1 - \frac{1}{|X||Y|} \leq 1 - \frac{1}{n^2},$$

so equality would have to hold throughout. This would force $|X||Y| = n^2$, hence $X = A$ and $Y = B$. But then $d(X, Y) = d(A, B)$, which does not give a deviation of $\frac{1}{2}$. Therefore (X, Y) is complete. Since $|X|, |Y| \geq \frac{n}{2}$, we may choose $\frac{n}{2}$ vertices from each side and obtain a copy of $K_{\frac{n}{2},\frac{n}{2}}$. □

Exercise 1.2.
(Chernoff Bound). Let $X_1, \ldots, X_n$ be independent indicator variables such that $\Pr[X_i = 1] = p$ for each $i \in [n]$. Let $X = \sum_{i=1}^{n} X_i$. We know that

$$\Pr[|X - \mathbb{E}(X)| > \alpha] \leq 2e^{-\frac{2\alpha^2}{n}}.$$

> Let $G(A, B, p)$ denote a random bipartite graph on two disjoint vertex sets A, B with $|A| = |B| = n$ such that for each $a \in A$ and $b \in B$, ab is chosen to be an edge with probability p independently.
> Prove: For any $\epsilon > 0$, $G(A, B, p)$ is ϵ-regular with probability tending to 1 as n tends to infinity.

Proof. Consider $0 < \epsilon < 1$, Set

$$\eta = \frac{\epsilon}{3}.$$

For fixed subsets $X \subseteq A$ and $Y \subseteq B$, the random variable $e(X, Y)$ is a sum of $|X||Y|$ independent indicator variables, and

$$\mathbb{E}[e(X, Y)] = p|X||Y|.$$

Applying the given Chernoff bound with $\alpha = \eta|X||Y|$, we get

$$\Pr\big[|d(X, Y) - p| > \eta\big] \leq 2e^{-2\eta^2|X||Y|}.$$

Now restrict to pairs (X, Y) satisfying

$$|X| \geq \epsilon n, \qquad |Y| \geq \epsilon n.$$

For each such pair, $|X||Y| \geq \epsilon^2 n^2$, and therefore

$$\Pr\big[|d(X, Y) - p| > \eta\big] \leq 2e^{-2\eta^2\epsilon^2 n^2}.$$

There are at most 2^n choices for X and at most 2^n choices for Y. Hence, by the union bound, the probability that some such pair satisfies $|d(X, Y) - p| > \eta$ is at most

$$2 \cdot 4^n e^{-2\eta^2\epsilon^2 n^2} = o(1).$$

Thus, with probability tending to 1, every pair (X, Y) with $|X|, |Y| \geq \epsilon n$ satisfies

$$|d(X, Y) - p| \leq \eta.$$

This also holds for the whole pair (A, B). Hence for every such X, Y,

$$|d(X, Y) - d(A, B)| \leq |d(X, Y) - p| + |d(A, B) - p| \leq 2\eta < \epsilon.$$

Therefore $G(A, B, p)$ is ϵ-regular with probability tending to 1. □