1 Assignments: June 2nd, 2026

> **Exercise 1.1.**
> Prove: Every Sperner-colored simplicial subdivision contains an odd number of rainbow cells. 

Proof. Let

$$\Delta = \operatorname{conv}(v_0, v_1, \dots, v_d),$$

where v_i has color i, and let $\mathcal{T}$ be a Sperner-colored simplicial subdivision of Δ. We use induction on d.

The case $d = 0$ is immediate. For $d = 1$, the subdivision is an interval whose endpoints have colors 0 and 1. Each rainbow edge is a place where the color changes. Since the colors at the two ends are different, the number of such changes is odd.

Assume now that $d \geq 2$ and that the result holds in dimension $d - 1$. Let F be the facet of Δ opposite v_d. The restriction of $\mathcal{T}$ to F is a Sperner-colored subdivision of a $(d-1)$-simplex with colors $0, 1, \dots, d-1$. By induction, it contains an odd number of rainbow $(d-1)$-cells.

Define a graph Γ whose vertices are the d-cells of $\mathcal{T}$, together with one extra vertex $*$. For every $(d-1)$-cell with color set

$$\{0, 1, \dots, d-1\},$$

add an edge to Γ. If the cell is shared by two d-cells, join the corresponding vertices. If it is a boundary cell, join its unique containing d-cell to $*$.

Every boundary cell counted above lies in F. Indeed, if it lies in the facet of Δ opposite v_j, then the Sperner condition allows only colors from $\{0, 1, \dots, d\} \setminus \{j\}$. Since the cell uses all the colors $0, 1, \dots, d-1$, we must have $j = d$. Therefore

$$\deg_\Gamma(*) \equiv 1 \pmod 2$$

by the induction hypothesis.

Let T be a d-cell of $\mathcal{T}$. Its degree in Γ is the number of its facets with color set $\{0, 1, \dots, d-1\}$.

If T is rainbow, exactly one facet is counted, namely the one obtained by deleting the vertex of color d. Thus $\deg_\Gamma(T) = 1$.

If T uses every color $0, 1, \dots, d-1$ but does not use color d, then exactly one of these colors is repeated among the $d+1$ vertices of T. Deleting either copy of the repeated color gives a counted facet, so $\deg_\Gamma(T) = 2$.

In all remaining cases, T misses at least one color from $0, 1, \dots, d-1$, so no facet is counted and $\deg_\Gamma(T) = 0$.

Thus the odd-degree vertices of Γ, other than $*$, are exactly the rainbow d-cells. By the handshaking lemma, Γ has an even number of odd-degree vertices. Since $*$ has odd degree, the number of rainbow d-cells is odd. $\square$