1 Assignments: June 9th, 2026

We first recall the form of Fisher's inequality used below.

remark. Maybe there is something wrong. It seems that what we need is Fisher's inequality ?

Theorem 1.1 (Fisher's inequality). *Let $\mathcal{A} = \{A_1, \ldots, A_m\}$ be a family of subsets of an N-element set. Suppose that, for some nonnegative integer λ,*

$$|A_i \cap A_j| = \lambda \quad \text{for all } i \neq j, \qquad |A_i| > \lambda \quad \text{for all } i.$$

Then $m \leq N$.

Exercise 1.1.
Apply Fisher's inequality to prove that n points in the plane either lie on one line or determine at least n distinct lines.

Proof. Let P be the set of the given points, and let $\mathcal{L}$ be the set of all lines determined by pairs of points in P. Suppose that the points are not all collinear.

For each point $x \in P$, let

$$\mathcal{L}_x = \{\ell \in \mathcal{L} : x \in \ell\}.$$

Every point lies on at least two lines in $\mathcal{L}$. Indeed, choose $y \in P \setminus \{x\}$. Since the points are not all on the line xy, there is a point $z \in P$ outside xy, and the lines xy and xz are distinct. Therefore $|\mathcal{L}_x| > 1$.

For two distinct points $x, y \in P$, the only line belonging to both $\mathcal{L}_x$ and $\mathcal{L}_y$ is the line xy. Hence

$$|\mathcal{L}_x \cap \mathcal{L}_y| = 1.$$

Applying Fisher's inequality to the family $\{\mathcal{L}_x : x \in P\}$ on the ground set $\mathcal{L}$, with $\lambda = 1$, gives

$$n = |P| \leq |\mathcal{L}|.$$

Thus the points determine at least n distinct lines. □

Exercise 1.2.
Let p be a prime and $L \subseteq \mathbb{Z}_p$. Suppose that $\mathcal{F} \subseteq 2^{[n]}$ satisfies

$$|A| \notin L \pmod p \quad \text{for every } A \in \mathcal{F},$$

and

$$|A \cap B| \in L \pmod p \quad \text{for every two distinct } A, B \in \mathcal{F}.$$

Prove that

$$|\mathcal{F}| \leq \sum_{k=0}^{|L|} \binom{n}{k}.$$

Proof. Put $s = |L|$ and work over the field $\mathbb{F}_p$. For each $A \in \mathcal{F}$, define

$$f_A(x_1, \ldots, x_n) = \prod_{\ell \in L} \left(\sum_{i \in A} x_i - \ell \right).$$

Let $\mathbf{1}_B \in \{0,1\}^n$ be the incidence vector of a set B. Then

$$f_A(\mathbf{1}_B) = \prod_{\ell \in L} (|A \cap B| - \ell).$$

If $A \neq B$, the assumption $|A \cap B| \in L \pmod{p}$ gives $f_A(\mathbf{1}_B) = 0$. On the other hand, since $|A| \notin L \pmod{p}$,

$$f_A(\mathbf{1}_A) \neq 0.$$

It follows that the functions f_A, restricted to the Boolean cube $\{0,1\}^n$, are linearly independent. To see this, suppose that

$$\sum_{A \in \mathcal{F}} c_A f_A = 0$$

on $\{0,1\}^n$. Evaluating at $\mathbf{1}_B$ leaves only the term $c_B f_B(\mathbf{1}_B)$, so $c_B = 0$ for every $B \in \mathcal{F}$.

Each f_A has degree at most s. On the Boolean cube, replacing every power x_i^r with $r \geq 1$ by x_i does not change the function. Hence every f_A is represented by a multilinear polynomial of degree at most s. The space of such polynomials has basis

$$\left\{ \prod_{i \in S} x_i : S \subseteq [n],\ |S| \leq s \right\}$$

and therefore has dimension

$$\sum_{k=0}^{s} \binom{n}{k}.$$

Since the functions $\{f_A : A \in \mathcal{F}\}$ are linearly independent, the upper bound follows. $\square$