1 Assignments: Mar 10th, 2026

Exercise 1.1. If $\binom{n}{k}p^{\binom{k}{2}} + \binom{n}{t}(1-p)^{\binom{t}{2}} < 1$ for some $p \in [0,1]$, then the Ramsey number $R(k,t) > n$.

Proof. We use the probabilistic method. Consider a random graph $G \sim G(n,p)$ on n vertices where each edge appears independently with probability p.

For any subset $S \subseteq V(G)$ with $|S| = k$, let A_S be the event that S induces a k-clique (complete subgraph on k vertices). The probability of A_S is:

$$\Pr[A_S] = p^{\binom{k}{2}}$$

since all $\binom{k}{2}$ edges within S must be present.

Similarly, for any subset $T \subseteq V(G)$ with $|T| = t$, let B_T be the event that T induces an independent set of size t. The probability of B_T is:

$$\Pr[B_T] = (1-p)^{\binom{t}{2}}$$

since all $\binom{t}{2}$ edges within T must be absent.

By the union bound, the probability that G contains either a k-clique or an independent set of size t is at most:

$$\sum_{S:|S|=k} \Pr[A_S] + \sum_{T:|T|=t} \Pr[B_T] = \binom{n}{k}p^{\binom{k}{2}} + \binom{n}{t}(1-p)^{\binom{t}{2}}$$

By assumption, this quantity is strictly less than 1. Therefore:

$$\Pr[G \text{ has no } k\text{-clique and no independent set of size } t] > 0$$

This means there exists at least one graph on n vertices with no k-clique and no independent set of size t. By definition of the Ramsey number, $R(k,t) > n$. □

Exercise 1.2. For any graph G with $2n$ vertices and m edges, G contains a bipartite subgraph with at least $\frac{mn}{2n-1}$ edges.

Proof. Let $G = (V,E)$ be a graph with $|V| = 2n$ and $|E| = m$. We use a probabilistic argument.

Consider a random bipartition of V into two sets A and B with $|A| = |B| = n$, chosen uniformly at random from all $\binom{2n}{n}$ such bipartitions.

For any edge $e = \{u,v\} \in E$, we compute the probability that e crosses the bipartition (i.e., $u \in A, v \in B$ or $u \in B, v \in A$).

Given edge $e = \{u,v\}$, we need to count bipartitions where u and v are in different parts. Fix $u \in A$ and $v \in B$. We need to choose $n-1$ more vertices for A from the remaining $2n-2$ vertices. The number of such bipartitions is $\binom{2n-2}{n-1}$.

Similarly, if $u \in B$ and $v \in A$, we also get $\binom{2n-2}{n-1}$ bipartitions.

Thus, the probability that edge e crosses the bipartition is:

$$\Pr[e \text{ crosses}] = \frac{2\binom{2n-2}{n-1}}{\binom{2n}{n}}$$

Simplifying:

$$\binom{2n}{n} = \frac{(2n)!}{n! \cdot n!} = \frac{2n \cdot (2n-1) \cdot (2n-2)!}{n \cdot (n-1)! \cdot n \cdot (n-1)!} = \frac{2n(2n-1)}{n^2}\binom{2n-2}{n-1} = \frac{2(2n-1)}{n}\binom{2n-2}{n-1}$$

Therefore:

$$\Pr[e \text{ crosses}] = \frac{2\binom{2n-2}{n-1}}{\frac{2(2n-1)}{n}\binom{2n-2}{n-1}} = \frac{n}{2n-1}$$

Let X be the random variable counting the number of crossing edges. By linearity of expectation:

$$\mathbb{E}[X] = \sum_{e \in E} \Pr[e \text{ crosses}] = m \cdot \frac{n}{2n-1} = \frac{mn}{2n-1}$$

Since $\mathbb{E}[X] = \frac{mn}{2n-1}$, there exists at least one bipartition with at least $\frac{mn}{2n-1}$ crossing edges. These crossing edges form a bipartite subgraph. $\square$

Exercise 1.3. Prove that Bollobás's Theorem implies Sperner's Theorem.

Proof. First, we state both theorems:

Theorem 1.1 (Sperner's Theorem, 1928)**.** *Let $\mathcal{F}$ be a family of subsets of $[n] = \{1, 2, \ldots, n\}$ such that no set in $\mathcal{F}$ contains another (i.e., $\mathcal{F}$ is an antichain under inclusion). Then:*

$$|\mathcal{F}| \le \binom{n}{\lfloor n/2 \rfloor}$$

Theorem 1.2 (Bollobás's Theorem, 1965)**.** *Let $A_1, \ldots, A_m$ and $B_1, \ldots, B_m$ be subsets of $[n]$ such that:*

(i) $A_i \cap B_i = \emptyset$ for all $i \in [m]$;

(ii) $A_i \cap B_j \neq \emptyset$ for all $i \neq j$.

Then:

$$\sum_{i=1}^{m} \frac{1}{\binom{|A_i|+|B_i|}{|A_i|}} \le 1$$

In particular, $m \le \binom{n}{\lfloor n/2 \rfloor}$.

Now we prove that Bollobás's Theorem implies Sperner's Theorem.

Let $\mathcal{F} = \{F_1, F_2, \ldots, F_m\}$ be an antichain on $[n]$. We will construct sets A_i and B_i satisfying the conditions of Bollobás's Theorem.

For each $i \in [m]$, define:

$$A_i = F_i \quad \text{and} \quad B_i = [n] \setminus F_i$$

We verify the conditions:

(i) $A_i \cap B_i = F_i \cap ([n] \setminus F_i) = \emptyset$ for all i. ✓

(ii) For $i \neq j$, we need $A_i \cap B_j \neq \emptyset$, i.e., $F_i \cap ([n] \setminus F_j) \neq \emptyset$.

This is equivalent to $F_i \not\subseteq F_j$. Since $\mathcal{F}$ is an antichain, we have $F_i \not\subseteq F_j$ for all $i \neq j$. ✓

Now apply Bollobás's Theorem. Note that $|A_i|+|B_i| = |F_i|+(n-|F_i|) = n$ for all i. Therefore:

$$\sum_{i=1}^{m} \frac{1}{\binom{n}{|F_i|}} \leq 1$$

Since $\binom{n}{k}$ is maximized at $k = \lfloor n/2 \rfloor$, we have $\binom{n}{|F_i|} \leq \binom{n}{\lfloor n/2 \rfloor}$ for all i, which implies:

$$\frac{1}{\binom{n}{|F_i|}} \geq \frac{1}{\binom{n}{\lfloor n/2 \rfloor}}$$

Therefore:

$$\frac{m}{\binom{n}{\lfloor n/2 \rfloor}} = \sum_{i=1}^{m} \frac{1}{\binom{n}{\lfloor n/2 \rfloor}} \leq \sum_{i=1}^{m} \frac{1}{\binom{n}{|F_i|}} \leq 1$$

This gives $m \leq \binom{n}{\lfloor n/2 \rfloor}$, which is exactly Sperner's Theorem. $\square$