1 Assignments: Mar 17th, 2026

Exercise 1.1.

(a) Prove: There is a 2-edge-coloring of K_n with at most $\binom{n}{k}2^{1-\binom{k}{2}}$ monochromatic K_k.

(b) Prove: $R(k,k) > n - \binom{n}{k}2^{1-\binom{k}{2}}$.

Proof. **(a)** Consider a random 2-edge-coloring of K_n where each edge is colored independently with red or blue, each with probability $\frac{1}{2}$. For any subset $S \subseteq V(K_n)$ with $|S| = k$, let X_S be the indicator random variable for the event that S induces a monochromatic K_k. The probability that all $\binom{k}{2}$ edges within S are the same color is:

$$\Pr[X_S = 1] = 2 \cdot \left(\frac{1}{2}\right)^{\binom{k}{2}} = 2^{1-\binom{k}{2}}$$

since there are 2 choices for the color (red or blue) and each edge has probability $\frac{1}{2}$ of being that color.

Let X be the total number of monochromatic K_k subgraphs. And then, we have:

$$\mathbb{E}[X] = \sum_{S:|S|=k} \mathbb{E}[X_S] = \binom{n}{k} \cdot 2^{1-\binom{k}{2}}$$

Since the expected number of monochromatic K_k is $\binom{n}{k}2^{1-\binom{k}{2}}$.In other word, there must exist at least one 2-edge-coloring of K_n with at most this many monochromatic K_k.

(b) Let $N = \binom{n}{k}2^{1-\binom{k}{2}}$. From (a) we know that there exists a 2-edge-coloring of K_n with at most N monochromatic K_k.

For each monochromatic K_k, we delete one vertex. Since there are at most N monochromatic K_k, we delete at most N vertices in total.

Let G' be the resulting graph on at least $n - N$ vertices. By construction, G' has no monochromatic K_k in either color, because we removed at least one vertex from every monochromatic K_k that existed in the original coloring.

Therefore, there exists a 2-edge-coloring of K_{n-N} with no monochromatic K_k, which implies:

$$R(k,k) > n - N = n - \binom{n}{k}2^{1-\binom{k}{2}}$$

□

Exercise 1.2. A tournament $T_n = (V, D)$ is an oriented complete graph, i.e., $V = [n]$ and for any two vertices $u, v \in V$, either $(u,v) \in D$ or $(v,u) \in D$. A Hamiltonian path of T_n is a sequence of vertices $i_1, i_2, \ldots, i_n$ such that all i_j's are distinct elements of $[n]$ and for each $k \in \{1, 2, \ldots, n-1\}$, $(i_k, i_{k+1}) \in D$.
Prove: There is a tournament T_n with at least $n!2^{-(n-1)}$ Hamiltonian paths.

Proof. Consider a random tournament T_n on vertex set $[n]$, where for each pair of distinct vertices $\{u, v\}$, the edge orientation is chosen independently and uniformly at random (i.e., $(u,v) \in D$ with probability $\frac{1}{2}$ and $(v,u) \in D$ with probability $\frac{1}{2}$).

For any permutation $\pi = (i_1, i_2, \ldots, i_n)$ of $[n]$, let X_π be the indicator random variable for the event that π forms a Hamiltonian path in T_n. That is:

$$X_\pi = \mathbf{1}_{\{(i_k, i_{k+1}) \in D \text{ for all } k=1,\ldots,n-1\}}$$

For π to be a Hamiltonian path, we need all $n-1$ directed edges $(i_1, i_2), (i_2, i_3), \ldots, (i_{n-1}, i_n)$ to be present in T_n. Since each edge orientation is chosen independently with probability $\frac{1}{2}$:

$$\Pr[X_\pi = 1] = \left(\frac{1}{2}\right)^{n-1} = 2^{-(n-1)}$$

Let X be the total number of Hamiltonian paths in T_n. We have:

$$\mathbb{E}[X] = \sum_{\pi \in S_n} \mathbb{E}[X_\pi] = n! \cdot 2^{-(n-1)}$$

where S_n denotes the set of all $n!$ permutations of $[n]$.

Since $\mathbb{E}[X] = n!2^{-(n-1)}$, there must exist at least one tournament T_n with at least $n!2^{-(n-1)}$ Hamiltonian paths. $\square$