1 Assignments: Mar 24th, 2026

Exercise 1.1.
Prove: For any two probabilities p_1 and p_2 such that $p_1 \leq p_2$ and for a monotone increasing graph property $\mathcal{Q}$, we have that

$$\mathbb{P}[G_{n,p_1} \in \mathcal{Q}] \leq \mathbb{P}[G_{n,p_2} \in \mathcal{Q}].$$

Proof. Let $G_1 \sim G_{n,p_1}$ and $G_2 \sim G_{n,p_2}$ be two random graphs on the same vertex set $V = \{1, 2, \ldots, n\}$. Since $p_1 \leq p_2$, we can write $p_2 = p_1 + (p_2 - p_1)$ where $0 \leq p_2 - p_1 \leq 1 - p_1$. For each potential edge $e \in \binom{V}{2}$, we generate independent random variables: $X_e \sim \text{Bernoulli}(p_1)$ and $Y_e \sim \text{Bernoulli}\left(\frac{p_2-p_1}{1-p_1}\right)$ independent of X_e, then we define $Z_e = \max\{X_e, Y_e\}$. Note that since $p_2 \leq 1$, we have $\frac{p_2-p_1}{1-p_1} \leq 1$.

First, $\mathbb{P}[X_e = 1] = p_1$, so the graph G_1 with edges $\{e : X_e = 1\}$ follows G_{n,p_1}. For Z_e, we have:

$$\begin{aligned}
\mathbb{P}[Z_e = 1] &= \mathbb{P}[X_e = 1 \text{ or } Y_e = 1] \\
&= 1 - \mathbb{P}[X_e = 0]\mathbb{P}[Y_e = 0] \\
&= 1 - (1 - p_1)\left(1 - \frac{p_2 - p_1}{1 - p_1}\right) \\
&= 1 - (1 - p_1) \cdot \frac{1 - p_1 - (p_2 - p_1)}{1 - p_1} \\
&= 1 - (1 - p_2) = p_2.
\end{aligned}$$

Thus the graph G_2 with edges $\{e : Z_e = 1\}$ follows G_{n,p_2}. By construction, $X_e \leq Z_e$ for all $e \in \binom{V}{2}$, which means $E(G_1) = \{e : X_e = 1\} \subseteq \{e : Z_e = 1\} = E(G_2)$. Therefore, G_1 is always a subgraph of G_2 in this coupling, i.e., $G_1 \subseteq G_2$. Since $\mathcal{Q}$ is a monotone increasing graph property, we have $G_1 \subseteq G_2$ and $G_1 \in \mathcal{Q}$ implies $G_2 \in \mathcal{Q}$, equivalently, the indicator satisfies $\mathbf{1}_{\{G_1 \in \mathcal{Q}\}} \leq \mathbf{1}_{\{G_2 \in \mathcal{Q}\}}$ almost surely. For expectations:

$$\mathbb{P}[G_1 \in \mathcal{Q}] = \mathbb{E}[\mathbf{1}_{\{G_1 \in \mathcal{Q}\}}] \leq \mathbb{E}[\mathbf{1}_{\{G_2 \in \mathcal{Q}\}}] = \mathbb{P}[G_2 \in \mathcal{Q}].$$

In other word:

$$\mathbb{P}[G_{n,p_1} \in \mathcal{Q}] \leq \mathbb{P}[G_{n,p_2} \in \mathcal{Q}]$$

□

Exercise 1.2.
Determine the threshold function $p^*(n)$ for a random graph G_{n,p_n} to contain a triangle, and prove it.

Proof. The threshold function for containing a triangle is $p^*(n) = \frac{1}{n}$. More precisely, if $p_n \ll \frac{1}{n}$ (i.e., $p_n = o(1/n)$), then $\mathbb{P}[G_{n,p_n}$ contains a triangle$] \to 0$ as $n \to \infty$; and if $p_n \gg \frac{1}{n}$ (i.e., $p_n = \omega(1/n)$), then $\mathbb{P}[G_{n,p_n}$ contains a triangle$] \to 1$ as $n \to \infty$.

Let X_n be the number of triangles in G_{n,p_n}. There are $\binom{n}{3}$ possible triangles (3-vertex subsets), each appearing with probability p_n^3 (all 3 edges must be present), so:

$$\mathbb{E}[X_n] = \binom{n}{3}p_n^3 = \frac{n(n-1)(n-2)}{6}p_n^3 \sim \frac{n^3 p_n^3}{6}.$$

When $p_n \ll 1/n$, we have $np_n = o(1)$, so $\mathbb{E}[X_n] \sim \frac{(np_n)^3}{6} \to 0$. By Markov's inequality, $\mathbb{P}[X_n \geq 1] \leq \mathbb{E}[X_n] \to 0$, thus $\mathbb{P}[G_{n,p_n}$ contains a triangle$] \to 0$.

When $p_n \gg 1/n$, we have $np_n \to \infty$, so $\mathbb{E}[X_n] \sim \frac{(np_n)^3}{6} \to \infty$. To show $\mathbb{P}[X_n \geq 1] \to 1$, we use Chebyshev's inequality :

$$\mathbb{P}[X_n = 0] \leq \frac{\mathrm{Var}(X_n)}{(\mathbb{E}[X_n])^2},$$

and we need to show $\frac{\mathrm{Var}(X_n)}{(\mathbb{E}[X_n])^2} \to 0$.

Let $T_1, T_2, \ldots, T_m$ be all possible triangles where $m = \binom{n}{3}$, and let I_i be the indicator that triangle T_i is present. Then $X_n = \sum_{i=1}^m I_i$, and:

$$\mathrm{Var}(X_n) = \sum_{i=1}^m \mathrm{Var}(I_i) + \sum_{i \neq j} \mathrm{Cov}(I_i, I_j).$$

For the diagonal terms, $\mathrm{Var}(I_i) = p_n^3(1 - p_n^3) \leq p_n^3$, so $\sum_i \mathrm{Var}(I_i) \leq mp_n^3 = \mathbb{E}[X_n]$. For the off-diagonal terms, two triangles T_i and T_j can share 0 edges (disjoint), 1 edge (forms a "diamond" with 4 vertices), or 2/3 edges (same triangle, excluded since $i \neq j$). For disjoint triangles ($|V(T_i) \cap V(T_j)| \leq 1$, so 0 shared edges), I_i and I_j are independent, so $\mathrm{Cov}(I_i, I_j) = 0$. For triangles sharing exactly one edge ($|V(T_i) \cap V(T_j)| = 2$), there are $O(n^4)$ such pairs (choose 4 vertices, then choose which edge is shared). For each such pair, both triangles appear if all 5 edges are present, so $\mathbb{E}[I_i I_j] = p_n^5$ and $\mathbb{E}[I_i]\mathbb{E}[I_j] = p_n^6$, giving $\mathrm{Cov}(I_i, I_j) = p_n^5 - p_n^6 = p_n^5(1 - p_n) \leq p_n^5$. The total contribution is $O(n^4 p_n^5)$.

Putting together the variance bound:

$$\mathrm{Var}(X_n) \leq \mathbb{E}[X_n] + O(n^4 p_n^5) = O(n^3 p_n^3 + n^4 p_n^5).$$

Now compute:

$$\frac{\mathrm{Var}(X_n)}{(\mathbb{E}[X_n])^2} \leq \frac{O(n^3 p_n^3 + n^4 p_n^5)}{(n^3 p_n^3)^2} = O\left(\frac{1}{n^3 p_n^3} + \frac{1}{n^2 p_n}\right).$$

When $p_n = \omega(1/n)$, we have $n^3 p_n^3 \to \infty$ so $\frac{1}{n^3 p_n^3} \to 0$, and $n^2 p_n \to \infty$ so $\frac{1}{n^2 p_n} \to 0$. Therefore, $\frac{\mathrm{Var}(X_n)}{(\mathbb{E}[X_n])^2} \to 0$, which implies $\mathbb{P}[X_n = 0] \to 0$, so $\mathbb{P}[G_{n,p_n}$ contains a triangle$] = \mathbb{P}[X_n \geq 1] \to 1$.

In conclusion, we have proven that $p^*(n) = 1/n$ is a threshold function for the property of containing a triangle:

$$\mathbb{P}[G_{n,p_n} \text{ contains a triangle}] \to \begin{cases} 0 & \text{if } p_n \ll 1/n, \\ 1 & \text{if } p_n \gg 1/n. \end{cases}$$

$\square$