1 Assignments: Mar 31st, 2026

Exercise 1.1.
Given n irrational numbers $x_1, \dots, x_n$, determine the maximum number of pairs (x_i, x_j) such that $x_i + x_j$ is rational.

Solution. The maximum is $\lfloor n/2 \rfloor$.

Partition $\{x_1, \dots, x_n\}$ by the equivalence relation $x_i \sim x_j$ if $x_i - x_j \in \mathbb{Q}$. Each class C has the form $C = \{r_C + q : q \in Q_C\}$ for some irrational r_C and $Q_C \subset \mathbb{Q}$. Fix a rational q. The pairs with sum q form a matching: if $x_i + x_j = q$ and $x_i + x_k = q$, then $x_j = x_k$. So at most $\lfloor n/2 \rfloor$ pairs sum to any fixed rational.

Suppose $x_i + x_j = q_1$ and $x_i + x_k = q_2$ with $q_1 \neq q_2$. Then $x_j - x_k = q_1 - q_2 \in \mathbb{Q}$, so $x_j \sim x_k$. If also $x_j + x_k = q_3 \in \mathbb{Q}$, then $2x_j = q_3 + q_1 - q_2 \in \mathbb{Q}$, contradiction. Thus within each equivalence class, all pairs with rational sum have the same sum value.

Hence all such pairs form a single matching, giving at most $\lfloor n/2 \rfloor$ pairs.

This bound is achieved by taking $\lfloor n/2 \rfloor$ distinct irrationals $a_1, \dots, a_{\lfloor n/2 \rfloor}$ and setting $x_i = a_i$, $x_{\lfloor n/2 \rfloor + i} = 1 - a_i$ (plus one extra irrational if n is odd). Then $x_i + x_{\lfloor n/2 \rfloor + i} = 1$ for each i. □

Exercise 1.2.
Prove: In the induction proof, $ex(n, K_{r+1})$ is achieved uniquely by the Turán graph $T_{n,r}$.

Proof. Induction on n. For $n \leq r$, we have $ex(n, K_{r+1}) = \binom{n}{2}$ achieved only by $K_n = T_{n,r}$. Let $n > r$ and let G be K_{r+1}-free with n vertices and $e(G) = ex(n, K_{r+1}) = e(T_{n,r})$. We show $G \cong T_{n,r}$.

Since G is extremal, it contains a K_r (otherwise we could add an edge). Let $A = V(K_r)$ and $B = V(G) \setminus A$. Since G is K_{r+1}-free, each vertex in B has at most $r - 1$ neighbors in A. Thus

$$e(G) = \binom{r}{2} + e(G[B]) + e(A, B) \leq \binom{r}{2} + ex(n - r, K_{r+1}) + (r - 1)(n - r).$$

By induction, $e(G[B]) \leq e(T_{n-r,r})$ with equality iff $G[B] \cong T_{n-r,r}$. Since $e(T_{n,r}) = \binom{r}{2} + e(T_{n-r,r}) + (r - 1)(n - r)$, equality $e(G) = e(T_{n,r})$ requires $G[B] \cong T_{n-r,r}$ and each vertex in B has exactly $r - 1$ neighbors in A.

Moreover, the non-neighbors in A must partition B so that G is complete r-partite with parts as equal as possible. This forces $G \cong T_{n,r}$. □

Exercise 1.3.
Prove: Every n-vertex graph with at least kn edges contains a path of length k.

Proof. Induction on n. For $n \leq k$, the condition $e(G) \geq kn$ is impossible since $e(G) \leq \binom{n}{2} < kn$.

Let $n > k$ and let G have n vertices with $e(G) \geq kn$. If G is disconnected, some component C with n' vertices has at least $kn' \cdot (n'/n) > kn'$ edges, and we apply induction to C. So assume G is connected.

If G has a vertex v with $\deg(v) \leq k$, then $G - v$ has $n - 1$ vertices and $e(G - v) \geq kn - k = k(n - 1)$ edges. By induction, $G - v$ contains a path of length k, hence so does G.

So $\delta(G) \geq k+1$. Let $P = v_0 v_1 \cdots v_m$ be a longest path. If $m < k$, then all neighbors of v_0 lie on P. Since $\deg(v_0) \geq k+1$, vertex v_0 has at least $k+1$ neighbors in $\{v_1, \ldots, v_m\}$. Let v_j be the neighbor with maximum index. Then $j \geq k+1 > m \geq j$, contradiction. Thus $m \geq k$. $\square$