1 Assignments: Apr 7th, 2026

Exercise 1.1.
Let $\pi_n(H) := \frac{ex(n,H)}{\binom{n}{2}}$ and let $\pi(H) := \lim_{n\to+\infty} \pi_n(H)$. Use the Probabilistic Method to prove that $\pi_n(H)$ is non-increasing as n increases.

Proof. Let G be an n-vertex H-free graph with $ex(n, H)$ edges. Pick a vertex $v \in V(G)$ uniformly at random and delete it. The resulting graph G' has $n-1$ vertices.

The expected number of edges in G' is

$$\mathbb{E}[e(G')] = e(G) - \frac{1}{n}\sum_{v\in V(G)} \deg(v) = e(G) - \frac{2e(G)}{n} = e(G)\cdot\frac{n-2}{n}.$$

Since G' is H-free, we have $e(G') \le ex(n-1, H)$. Taking expectations,

$$ex(n,H)\cdot\frac{n-2}{n} \le ex(n-1,H).$$

Rearranging,

$$\frac{ex(n,H)}{\binom{n}{2}} = \frac{ex(n,H)}{\frac{n(n-1)}{2}} \le \frac{ex(n-1,H)}{\frac{(n-1)(n-2)}{2}} = \frac{ex(n-1,H)}{\binom{n-1}{2}}.$$

Thus $\pi_n(H) \le \pi_{n-1}(H)$, so $\pi_n(H)$ is non-increasing in n. □

Exercise 1.2.
Let X be a set of n points on a circle of radius 1. Then the number of pairs of distinct points in X whose Euclidean distance exceeds $\sqrt{3}$ is at most $\left\lfloor \frac{n^2}{4} \right\rfloor$. Moreover, this bound is tight.

Proof. Two points on a unit circle have distance greater than $\sqrt{3}$ if and only if the central angle between them exceeds $2\pi/3$ (since $d = 2\sin(\theta/2) > \sqrt{3}$ implies $\theta > 2\pi/3$).

Define a graph G on vertex set X where two points are adjacent if their distance exceeds $\sqrt{3}$. We claim G is triangle-free. Indeed, three points with pairwise distances all exceeding $\sqrt{3}$ would have pairwise central angles all exceeding $2\pi/3$, but $3\cdot(2\pi/3) = 2\pi$, so such a configuration is impossible on a circle.

By Turán's theorem, $e(G) \le \lfloor n^2/4 \rfloor$, with equality if and only if G is a complete bipartite graph with parts as equal as possible.

Then we claim this bound is tight. Let $\varepsilon \in (0, \pi/6)$. Place $\lfloor n/2 \rfloor$ points on the short arc with angles in $[-\varepsilon, \varepsilon]$, and place the remaining $\lceil n/2 \rceil$ points on the opposite short arc with angles in $[\pi-\varepsilon, \pi+\varepsilon]$. For any two points chosen from different arcs, the central angle between them is at least $\pi - 2\varepsilon > 2\pi/3$, hence their distance is greater than $\sqrt{3}$. Therefore every cross pair is counted, giving exactly

$$\lfloor n/2 \rfloor \cdot \lceil n/2 \rceil = \left\lfloor \frac{n^2}{4} \right\rfloor,$$

so the upper bound is attained. □