1 Assignments: Apr 21st, 2026

1.1 Review

Theorem 1.1 (Erdős–Renyi–Sői,1966;Brown,1966;Balogh–Joself,2010)**.**

$$(\frac{1}{2} - o(1))n^{\frac{3}{2}} \le ex(n, C_4) \le \frac{1}{4}(1 + (4n-3)^{\frac{1}{2}})n^{\frac{3}{2}} = (\frac{1}{2} + o(1))n^{\frac{3}{2}} \tag{1}$$

Definition 1.1 (labeled fork)**.** *Let F be the set of labeled forks,*

$$F = \{(u, v, w) | uv, uw \in E(G), v \neq w\}$$

*remark. One "labeled" point connects two different point on the graph.Just like a **fork**.*

Theorem 1.2 (Kővári–Sós–Turán theorem)**.** *For $\forall t \ge s \ge 2$,*

$$ex(n, K_{s,t}) \le \frac{1}{2}(t-1)^{\frac{1}{2}} n^{2-\frac{1}{s}} + \frac{1}{2}(s-1)n = O(n^{1-\frac{1}{s}}) \tag{2}$$

remark. Last definition is BFS Tree Method, which is systematic algorithm for exploring every node in a tree data structure level by level.

1.2 Homework

Exercise 1.1.
Prove: Every n-vertex graph with at least $2ex(n, C_4)$ edges contains at least $(\frac{3}{2} - o(1))n^2$ many copies of C_4.

Proof. Let G be an n-vertex graph with $m = e(G) \ge 2ex(n, C_4)$. For distinct vertices x, y, we note

$$N_{xy} := |N(x) \cap N(y)|.$$

Define a labeled fork to be an ordered triple (x, z, y) with $x \neq y$ and $xz, zy \in E(G)$. Let F be the number of labeled forks.

Counting by the center z:

$$F = \sum_{z \in V(G)} d(z)(d(z) - 1).$$

By Jensen Inequality,

$$F \ge n\left(\frac{2m}{n}\right)\left(\frac{2m}{n} - 1\right) = \frac{4m^2}{n} - 2m.$$

By theorem 1.1, we get

$$m \ge 2ex(n, C_4) = (1 + o(1))n^{3/2},$$

hence

$$F \ge (4 - o(1))n^2.$$

Counting forks by endpoints:

$$F = \sum_{x \neq y} N_{xy} = 2 \sum_{\{x,y\} \in \binom{V}{2}} N_{xy},$$

so

$$\sum_{\{x,y\}} N_{xy} \geq (2 - o(1))n^2.$$

Let $N = \binom{n}{2} = (\frac{1}{2} + o(1))n^2$. By convexity of $u \mapsto \binom{u}{2}$,

$$\sum_{\{x,y\}} \binom{N_{xy}}{2} \geq N \binom{\frac{1}{N} \sum_{\{x,y\}} N_{xy}}{2} \geq N \binom{4 - o(1)}{2} = (3 - o(1))n^2.$$

Now $\sum_{\{x,y\}} \binom{N_{xy}}{2}$ counts each C_4 exactly twice (once for each pair of opposite vertices). Therefore

$$\#C_4(G) \geq \frac{1}{2}(3 - o(1))n^2 = \left(\frac{3}{2} - o(1)\right) n^2.$$

$\square$

Exercise 1.2.

Prove: Every set of n points in the Euclidean plane $\mathbb{R}^2$ has at most $O(n^{3/2})$ unit distances. 

Proof. Let P be the n-point set, and build a graph G on vertex set P by joining two points iff their Euclidean distance is 1. Then $e(G)$ is exactly the number of unit distances.

We claim G is $K_{2,3}$-free. Indeed, if two points a, b had three common neighbors x_1, x_2, x_3, then each x_i would lie on both unit circles centered at a and b. But two distinct circles intersect in at most two points, contradiction.

Hence $e(G) \leq ex(n, K_{2,3})$. By theorem 1.2,

$$ex(n, K_{2,3}) \leq (3-1)^{1/2} n^{3/2} + (2-1)n = \sqrt{2}\, n^{3/2} + n = O(n^{3/2}).$$

Therefore the number of unit distances is $O(n^{3/2})$. $\square$

Exercise 1.3.

Prove: For fixed $k \geq 2$, there exists a constant c_k such that

$$ex\big(n, \mathcal{C}^e_{\leq 2k}\big) \leq c_k n^{1+1/k}, \qquad \mathcal{C}^e_{\leq 2k} = \{C_4, C_6, \dots, C_{2k}\}.$$

Proof. Let G be an n-vertex graph with no even cycle of length at most $2k$, and let $m = e(G)$.

Take a bipartite subgraph $H \subseteq G$ with at least $m/2$ edges. Then H is still $\mathcal{C}^e_{\leq 2k}$-free. Since H is bipartite, all its cycles are even, so in fact H has no cycle of length $\leq 2k$, i.e. girth$(H) > 2k$.

Let $n_H = |V(H)| \leq n$, and $d = 2e(H)/n_H$ be the average degree of H. By repeatedly deleting vertices of degree $< d/2$, we get a nonempty subgraph $H' \subseteq H$ with minimum degree

$$\delta(H') \geq d/2.$$

Also girth$(H') > 2k$.

Pick $v \in V(H')$ and run BFS: levels $L_0 = \{v\}, L_1, \ldots, L_k$. Because girth is $> 2k$, the ball of radius k is tree-like, hence

$$|L_1| \geq \delta, \qquad |L_i| \geq (\delta - 1)|L_{i-1}| \ (2 \leq i \leq k),$$

where $\delta = \delta(H')$. Therefore

$$n \geq n_H \geq |V(H')| \geq 1 + \delta \sum_{i=0}^{k-1} (\delta - 1)^i \geq (\delta - 1)^k.$$

So $\delta \leq n^{1/k} + 1$, and thus

$$d \leq 2\delta \leq 2n^{1/k} + 2.$$

Hence

$$e(H) = \frac{dn_H}{2} \leq \frac{dn}{2} \leq n^{1+1/k} + n.$$

Finally,

$$m \leq 2e(H) \leq 2n^{1+1/k} + 2n \leq 4n^{1+1/k}.$$

So we may take $c_k = 4$ and has

$$ex\big(n, \mathcal{C}^e_{\leq 2k}\big) \leq c_k n^{1+1/k}.$$

$\square$