1 Assignments: Apr 28th, 2026

Exercise 1.1.
The odd-girth of a graph is the length of its shortest odd cycle. For any subset $U \subset V(G)$, let $G[U]$ denote the subgraph of G induced on U.
Let G be a graph of odd-girth at least $2k+3$. Assume that for any independent set I of G, $G - I$ contains an odd cycle.
Let A and B be two disjoint independent sets of G with $|A \cup B|$ maximized. Let C be an odd cycle in $G - A$. Choose a vertex $v \in V(C)$ and define a BFS tree rooted at v in G. Let

$$L_i = \{u \in V(G) : \mathrm{dist}(u, v) = i\}, \qquad L_0 = \{v\}.$$

For each $1 \le i \le k$, let

$$J_i := \left(\bigcup_{0 \le j < i} L_j\right) \cup (L_i \cap B) \cup \left(\bigcup_{j > i} (L_j \cap (A \cup B))\right).$$

Prove the following claims.

1. Each L_i is an independent set for each $i \le k$.
2. The subgraph $G[J_i]$ is bipartite.
3. $|A \cup B| \ge |J_i|$ and $|J_i \setminus (A \cup B)| \ge i - 1$.
4. $|L_i \cap A| \ge i - 1$ and thus $|A| \ge \frac{k(k-1)}{2}$.
5. The graph G has at least k^2 vertices.

Before our *Proof* , we have the following idea:

Idea. The BFS tree only concerns the component which contains v. Denote this component by R. When we compare J_i with $A \cup B$, we may do the comparison inside R: if $A \cap R$ and $B \cap R$ were not maximal in R, then we could improve A and B in the whole graph by changing them only in this component.

The main point is that an edge inside a low BFS level would create a short odd cycle. Hence the first k levels behave like the levels of a bipartite graph. The set J_i is built by replacing $L_i \cap A$ with some earlier vertices. Maximality of $A \cup B$ then forces $L_i \cap A$ to be large.

Proof. (1) Fix $i \le k$. Suppose that there is an edge xy with $x, y \in L_i$. Let P_x and P_y be shortest paths from v to x and y, respectively. Let w be the last common vertex of these two paths. Then the two subpaths from w to x and from w to y, together with the edge xy, form an odd cycle. Its length is at most

$$i + i + 1 = 2i + 1 \le 2k + 1,$$

contradicting that the odd-girth of G is at least $2k+3$. Therefore no such edge exists, and L_i is independent.

(2) In a BFS layering, every edge joins vertices either in the same level or in two consecutive levels. By (1), there are no edges inside $L_0, L_1, \ldots, L_k$.

Fix $1 \le i \le k$. Define

$$X = \left(\bigcup_{\substack{0 \le j < i \\ j \equiv i \pmod 2}} L_j \right) \cup \left(B \cap \bigcup_{j \ge i} L_j \right),$$

and

$$Y = \left(\bigcup_{\substack{0 \le j < i \\ j \not\equiv i \pmod 2}} L_j \right) \cup \left(A \cap \bigcup_{j > i} L_j \right).$$

Then $J_i = X \cup Y$. On the levels $L_0, \ldots, L_{i-1}$, this is just the usual parity coloring of BFS levels, so edges between these levels go from X to Y. For the vertices after level i, we put the vertices of B in X and the vertices of A in Y; this causes no edge inside one part since both A and B are independent. Finally, any edge from the earlier levels to $L_i \cap B$ must come from L_{i-1}, which lies in Y. Hence every edge of $G[J_i]$ goes between X and Y. So $G[J_i]$ is bipartite.

(3) Since $G[J_i]$ is bipartite, its two color classes form two disjoint independent sets. By the maximality of A and B,

$$|A \cup B| \ge |J_i|.$$

It remains to show that J_i contains at least $i-1$ vertices outside $A \cup B$. Since $C \subseteq G - A$, no vertex of C belongs to A. Starting from v, take the $i-1$ vertices before v and the $i-1$ vertices after v on the cycle C. Since $|C| \ge 2k+3 > 2i-1$, these vertices together with v form a path on $2i-1$ distinct vertices. Each of them has distance at most $i-1$ from v, hence lies in

$$\bigcup_{0 \le j < i} L_j \subseteq J_i.$$

On a path with $2i-1$ vertices, an independent set can contain at most i vertices. Since B is independent, at most i of the above vertices lie in B. Thus at least $(2i-1) - i = i-1$ of them lie in neither A nor B. Therefore

$$|J_i \setminus (A \cup B)| \ge i - 1.$$

(4) Now compare J_i with $(A \cap R) \cup (B \cap R)$. By the definition of J_i, among the vertices of $(A \cap R) \cup (B \cap R)$, the only ones not included in J_i are precisely the vertices of $L_i \cap A$. Thus

$$((A \cap R) \cup (B \cap R)) \setminus J_i = L_i \cap A.$$

Since J_i is bipartite by (2), its two color classes are two disjoint independent sets in R. By the maximality of $(A \cap R) \cup (B \cap R)$ inside R,

$$|(A \cap R) \cup (B \cap R)| \ge |J_i|.$$

Using

$$|S| - |T| = |S \setminus T| - |T \setminus S|$$

with $S = (A \cap R) \cup (B \cap R)$ and $T = J_i$, we get

$$|L_i \cap A| = |((A \cap R) \cup (B \cap R)) \setminus J_i| \ge |J_i \setminus (A \cup B)| \ge i - 1.$$

Since the BFS levels are disjoint, we can sum these inequalities over $i = 1, \dots, k$:

$$|A| \geq \sum_{i=1}^{k} |L_i \cap A| \geq \sum_{i=1}^{k} (i-1) = \frac{k(k-1)}{2}.$$

(5) The same argument can be applied with the roles of A and B interchanged. Indeed, B is independent, so by assumption $G - B$ contains an odd cycle. Repeating (1)–(4) with A and B switched gives

$$|B| \geq \frac{k(k-1)}{2}.$$

Moreover, $C \subseteq G - A$ has length at least $2k+3$. Since B is independent, B contains at most $k+1$ vertices of this odd cycle. Thus at least $k+2$ vertices of C are outside B. They are also outside A, because $C \subseteq G - A$.

Therefore

$$|V(G)| \geq |A| + |B| + k + 2 \geq k(k-1) + k + 2 = k^2 + 2 \geq k^2.$$

$\square$