1 Assignments: May 9th, 2026

Exercise 1.1.
Prove: For $k \geq 2$, every non-bipartite graph with

$$\delta(G) > \frac{2}{2k+1}n$$

contains an odd cycle of length at most $2k-1$.

Proof. Suppose, for contradiction, that G has no odd cycle of length at most $2k-1$. Since G is non-bipartite, it contains an odd cycle. Let C be a shortest odd cycle in G, and write $|C| = g$. Then $g \geq 2k+1$.

The cycle C has no chord. Indeed, any chord of C would split C into two cycles, one of which is an odd cycle shorter than C, contradicting the choice of C.

We claim that every vertex outside C has at most two neighbors on C. If some vertex $x \notin V(C)$ had three neighbors on C, these three neighbors would divide C into three arcs. Since C has odd length, at least one of these arcs has odd length. Together with the two edges from x to the endpoints of this arc, we obtain an odd cycle shorter than C (if the arc has length 1, this is just a triangle). This again contradicts the minimality of C.

Now count the degrees of vertices on C. Since each vertex of C has degree at least $\delta(G)$,

$$g\delta(G) \leq \sum_{v \in V(C)} d(v).$$

On the other hand, C contributes exactly $2g$ to this degree sum, and every vertex outside C contributes at most 2 edges into C. Hence

$$\sum_{v \in V(C)} d(v) \leq 2g + 2(n-g) = 2n.$$

Therefore

$$g\delta(G) \leq 2n.$$

Using $\delta(G) > \frac{2}{2k+1}n$, we get

$$g < 2k+1.$$

This contradicts $g \geq 2k+1$. Thus G must contain an odd cycle of length at most $2k-1$. □

Exercise 1.2.
Prove: Let H be a graph consisting of an 8-cycle

$$C = v_1 v_2 \cdots v_8 v_1$$

with chords v_1v_5 and v_2v_6. If a triangle-free graph G contains H as a subgraph, then

$$\delta(G) \leq \frac{3}{8}n.$$

Proof. Let $S = \{v_1, v_2, \ldots, v_8\}$ be the vertex set of the copy of H in G.

First observe that $\alpha(H) = 3$. Indeed, an independent set of size 4 in the 8-cycle must be one of the two alternating sets

$$\{v_1, v_3, v_5, v_7\}, \qquad \{v_2, v_4, v_6, v_8\}.$$

But the first set contains the chord v_1v_5, and the second contains the chord v_2v_6. Hence no independent set in H has size 4.

Since G is triangle-free, for every vertex $x \in V(G)$, the set $N_G(x) \cap S$ must be independent in H. Otherwise, if two vertices in $N_G(x) \cap S$ were adjacent in H, then together with x they would form a triangle in G. Therefore

$$|N_G(x) \cap S| \leq \alpha(H) = 3$$

for every $x \in V(G)$.

Now count the edges from S to the whole graph, counting an edge inside S twice. On one hand,

$$\sum_{i=1}^{8} d_G(v_i) \geq 8\delta(G).$$

On the other hand,

$$\sum_{i=1}^{8} d_G(v_i) = \sum_{x \in V(G)} |N_G(x) \cap S| \leq 3n.$$

Combining the two inequalities gives

$$8\delta(G) \leq 3n,$$

so

$$\delta(G) \leq \frac{3}{8}n.$$

□