作业 9

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问题 (1). 证明：对于方程

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} - a^2 \cdot \frac{\partial^2 u}{\partial x^2} = f(x,t), -\infty < x < +\infty \\ u|_{t=0} = 0, \quad \frac{\partial u}{\partial t}|_{t=0} = 0 \end{cases} \tag{1}$$

若 $w(x,t,\tau)$ 为如下方程的解：

$$\begin{cases} \frac{\partial^2 w}{\partial t^2} - a^2 \cdot \frac{\partial^2 w}{\partial x^2} = 0, t > \tau \\ w(x,t,\tau)|_{t=\tau} = 0 \\ \frac{\partial w}{\partial t}(x,t,\tau)|_{t=\tau} = f(x,\tau) \end{cases} \tag{2}$$

则 (1) 的解为：

$$u(x,t) = \int_0^t w(x,t,\tau)d\tau$$

注. 这实际上的证明就是 Duhamel 原理的证明.

证明. 我们只需要验证由 $u(x,t) = \int_0^t w(x,t,\tau)d\tau$ 定义的函数确实满足原方程 (1).

1. 当 $t = 0$ 时，积分区间退化为一点，故：

$$u(x,0) = \int_0^0 w(x,0,\tau)d\tau = 0$$

2. 我们对 u 关于 t 求偏导有：

$$\frac{\partial u}{\partial t}(x,t) = w(x,t,t) + \int_0^t \frac{\partial w}{\partial t}(x,t,\tau)d\tau$$

结合 w 的初始条件 $w(x,t,\tau)|_{t=\tau} = 0$，特别地当 $\tau = t$ 时有 $w(x,t,t) = 0$，所以：

$$\frac{\partial u}{\partial t}(x,t) = \int_0^t \frac{\partial w}{\partial t}(x,t,\tau)d\tau$$

再次验证初始条件：

$$\frac{\partial u}{\partial t}(x,0) = \int_0^0 \frac{\partial w}{\partial t}(x,0,\tau)d\tau = 0$$

我们对 $\frac{\partial u}{\partial t}$ 再次求导：

$$\frac{\partial^2 u}{\partial t^2}(x,t) = \frac{\partial w}{\partial t}(x,t,t) + \int_0^t \frac{\partial^2 w}{\partial t^2}(x,t,\tau)d\tau$$

由 w 的初始条件 $\frac{\partial w}{\partial t}(x,t,\tau)|_{t=\tau}=f(x,\tau)$，特别地当 $\tau=t$ 时有 $\frac{\partial w}{\partial t}(x,t,t)=f(x,t)$，所以:

$$\frac{\partial^2 u}{\partial t^2}(x,t)=f(x,t)+\int_0^t \frac{\partial^2 w}{\partial t^2}(x,t,\tau)d\tau$$

3. 接下来我们计算 u 关于 x 的偏导数:

$$\frac{\partial^2 u}{\partial x^2}(x,t)=\int_0^t \frac{\partial^2 w}{\partial x^2}(x,t,\tau)d\tau$$

故:

$$\begin{aligned}\frac{\partial^2 u}{\partial t^2}-a^2\cdot\frac{\partial^2 u}{\partial x^2}&=\left[f(x,t)+\int_0^t \frac{\partial^2 w}{\partial t^2}(x,t,\tau)d\tau\right]-a^2\int_0^t \frac{\partial^2 w}{\partial x^2}(x,t,\tau)d\tau\\&=f(x,t)+\int_0^t\left[\frac{\partial^2 w}{\partial t^2}(x,t,\tau)-a^2\cdot\frac{\partial^2 w}{\partial x^2}(x,t,\tau)\right]d\tau\end{aligned}$$

由于 $w(x,t,\tau)$ 在 $t>\tau$ 时满足齐次波动方程:

$$\frac{\partial^2 w}{\partial t^2}(x,t,\tau)-a^2\cdot\frac{\partial^2 w}{\partial x^2}(x,t,\tau)=0$$

因此:

$$\frac{\partial^2 u}{\partial t^2}-a^2\cdot\frac{\partial^2 u}{\partial x^2}=f(x,t)+\int_0^t 0\cdot d\tau=f(x,t)$$

综上所述，我们有:

$$\begin{cases}u(x,0)=0\\ \frac{\partial u}{\partial t}(x,0)=0\\ \frac{\partial^2 u}{\partial t^2}-a^2\cdot\frac{\partial^2 u}{\partial x^2}=f(x,t)\end{cases}$$

因此 $u(x,t)=\displaystyle\int_0^t w(x,t,\tau)d\tau$ 确实是方程 (1) 的解. □