Triangle-Intersecting Families

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Full Notes

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1 Section 1 Notes

1.1 Introduction

Background Statement (Original). *A basic theme in the field of extremal combinatorics is the study of the largest size of a structure given some combinatorial information concerning it.*

remark. 先给约束（例如“任意两集合交集非空”），再求能达到的最大规模。它不是在算一个固定对象，而是在所有满足条件的对象里做极值搜索。后文所有定理都围绕这个“约束 → 极值 → 结构”的主线展开。

Definition 1.1 (Definition 1.1 (Original)). *A family of graphs $\mathcal{F}$ is triangle-intersecting if for every $G, H \in \mathcal{F}$, $G \cap H$ contains a triangle.*

remark. 这里的“*intersecting*”不是说两个图有公共边就够了，而是要求公共部分里必须完整地出现一个三角形。这个条件比“交集非空”强很多，因此真正可行的范围会明显变小，也更容易出现清晰的极值结构。

Question 1 (Simonovits–Sós, Original). What is the maximum size of a triangle-intersecting family of subgraphs of the complete graph on n vertices?

remark. 这就是全文核心目标：在 K_n 的全部子图空间（规模是 $2^{\binom{n}{2}}$）里，找出满足定义 *1.1* 的最大子族，并刻画何时达到上界。

Theorem 1.1 (Theorem 1.2 (Original)). *Let $\mathcal{F}$ be a triangle-intersecting family of graphs on n vertices. Then*

$$|\mathcal{F}| \leq \frac{1}{8} 2^{\binom{n}{2}}.$$

Equality holds if and only if $\mathcal{F}$ consists of all graphs containing a fixed triangle.

remark. 这个结论包含两层信息。第一层是数值上界 1/8；第二层是结构唯一性：恰好达到上界的家族必须是“固定某个三角形，然后取所有包含它的图”。

Main Strengthenings (Original). *Our main result in this paper is actually a strengthening of the above in several aspects:*

- *odd-cycle-intersecting replaces triangle-intersecting;*
- *size is measured under product measure μ_p ($p \leq 1/2$), not only uniform measure;*
- *for $p = 1/2$, one may pass from intersecting to agreeing.*

remark. 第一步把禁忌结构从三角形放宽到奇环；第二步把计数问题推广成概率测度问题；第三步在均匀测度下把交集条件改成对称差条件。每一步都让问题更一般化。

1.2 Notation And Main Theorems

Definition 1.2 (Definition 1.3 (Original)). *A family $\mathcal{F}$ of subgraphs of K_n is triangle-intersecting (respectively odd-cycle-intersecting) if for every $G, H \in \mathcal{F}$, $G \cap H$ contains a triangle (respectively an odd cycle). We will say that $\mathcal{F}$ is triangle-agreeing (respectively odd-cycle-agreeing) if for every $G, H \in \mathcal{F}$, $G \oplus H$ contains a triangle (respectively an odd cycle).*

remark. 这里最关键的是两个运算的角色不同：$G \cap H$ 是共同保留的边，$G \oplus H$ 是两图不一致的边。前者描述"共同拥有什么"，后者描述"差异里缺了什么"。论文在两个视角间来回切换，是因为谱方法在群结构（对称差）下更自然。

Immediate Consequence (Original). *Note that $G \cap H$ is contained in $\overline{G \oplus H}$, so a triangle-intersecting family is also triangle-agreeing.*

remark. 因此 *intersecting* 条件蕴含 *agreeing* 条件，说明前者更"硬"，后者更"松"。证明时若先拿到更松条件下的 *sharp* 上界，通常也能回推更强条件版本。

难点（对应此处）：$G \cap H$ 与 $G \oplus H$ 为什么能在同一框架里处理？

remark. 关键在于后面把图空间看成 $\mathbb{Z}_2$ 上的阿贝尔群，$\oplus$ 成为天然运算；很多算子和谱分解都围绕这个运算定义。于是 *agreeing* 版本更容易进入代数分析，再通过包含关系或 *shift* 结果转回 *intersecting* 版本。

Basic Notation (Original). Let n be fixed. The power set of X is denoted by $P(X)$; $[n] = \{1, \ldots, n\}$; and

$$[n]^{(k)} = \{S \subseteq [n] : |S| = k\}.$$

It is convenient to identify the set of all subgraphs of K_n with the Abelian group

$$\mathbb{Z}_2^{[n]^{(2)}},$$

whose group operation is symmetric difference (denoted by $\oplus$ or $\triangle$).

remark. 这一段是后续所有计算的坐标系：把"图"改写成"边指示向量"，把"翻边"改写成"模 2 加法"。

Further Notation (Original). Further notation in the original text:

- $\overline{G}$ denotes the complement graph of G;
- $|G|$ is the number of edges of G;
- $v(G)$ is the number of non-isolated vertices of G;
- $G \cong H$ denotes graph isomorphism;
- if G is the disjoint union of G_1, G_2, write $G = G_1 \sqcup G_2$;
- $\emptyset$ denotes the empty graph on n vertices;

- a k-forest is any forest with k edges;
- K_4^- denotes the graph on four vertices with five edges;
- a biconnected component of G means a maximal biconnected subgraph of G.

Measure and Inner Product (Original). For any $p \in [0,1]$ and graph G on n vertices,

$$\mu_p(G) = p^{|G|}(1-p)^{\binom{n}{2}-|G|}.$$

For a family $\mathcal{F}$,

$$\mu_p(\mathcal{F}) = \sum_{G \in \mathcal{F}} \mu_p(G).$$

For functions $f, g : \mathbb{Z}_2^{\binom{n}{2}} \to \mathbb{R}$,

$$\langle f, g \rangle = \mathbb{E}(fg) = \sum_G \mu(G) f(G) g(G).$$

Additional Conventions (Original). Additional original conventions:

- identify $P(X)$ with $\{0,1\}^X = \mathbb{Z}_2^X$;
- an up-set $\mathcal{F}$ means: $S \in \mathcal{F}, T \supset S \Rightarrow T \in \mathcal{F}$;
- 1_P denotes the indicator of predicate P;
- for Abelian group A and $Y \subset A$, $\mathrm{Cay}(A,Y)$ is the Cayley graph with edges $\{\{a, a+y\} : a \in A, y \in Y\}$.

remark. 这些对象在方法段会立刻用到：尤其是 *Cayley* 图与特征值分析直接相关。

Terminology (Original). A triangle junta is a family of all subgraphs of K_n with a prescribed intersection with a given triangle. In the special case of the triangle junta being the family of all graphs containing a given triangle, we will call this family a $\triangle$umvirate.

remark. “*junta*” 表示成员资格由很少的坐标决定（只根据图在某个固定的三角形航的边集情况来决定这个图是否属于某个图族）。在这里，这些坐标就是固定三角形的三条边；而 $\triangle$*umvirate*（包含某个固定三角形的图）是最关键特例：三条边都必须在图里。它会反复作为极值构造出现。

Theorem 1.2 (Theorem 1.4 (Original)). ***[Extremal families]*** *Let $p \le 1/2$, and let $\mathcal{F}$ be an odd-cycle-intersecting family of subgraphs of K_n. Then $\mu_p(\mathcal{F}) \le p^3$, with equality if and only if $\mathcal{F}$ is a $\triangle$umvirate. Furthermore, in the case $p = 1/2$, if $\mathcal{F}$ is odd-cycle-agreeing then $\mu(\mathcal{F}) \le 1/8$, with equality if and only if $\mathcal{F}$ is a triangle junta.*
[Stability] *For each $p \le 1/2$ there exists a constant c_p (bounded for $p \in [\delta, 1/2]$, for any fixed $\delta > 0$) such that for any $\varepsilon \ge 0$, if $\mathcal{F}$ is an odd-cycle-intersecting family with $\mu_p(\mathcal{F}) \ge p^3 - \varepsilon$ then there exists a $\triangle$umvirate $\mathcal{T}$ such that*

$$\mu_p(\mathcal{T} \triangle \mathcal{F}) \le c_p \varepsilon.$$

For $p = 1/2$, the corresponding statement holds for odd-cycle-agreeing families.

remark. 这一条是全文主定理：上界、等号情形、稳定性一次给全。稳定性特别重要，因为它说明“几乎最优”的家族不会长得很奇怪，而是必须在测度意义上贴近某个 $\triangle$*umvirate*。

Corollary 1.3 (Corollary 1.5 (Original))**.** *Let $\alpha < 1/2$ and let $M = \alpha\binom{n}{2}$. Let $\mathcal{F}$ be an odd-cycle-intersecting family of graphs on n vertices with M edges each. Then*

$$|\mathcal{F}| \leq \binom{\binom{n}{2}-3}{M-3}.$$

Equality holds if and only if $\mathcal{F}$ is the set of all graphs with M edges containing a fixed triangle.

$$\textit{Moreover, if } |\mathcal{F}| > (1-\varepsilon)\binom{\binom{n}{2}-3}{M-3},$$

then there exists a triangle T such that all but at most $c\varepsilon|\mathcal{F}|$ of graphs in $\mathcal{F}$ contain T, where $c = c(\alpha)$.

remark. 这说明结论不仅在全体子图空间里成立，在“恰有 M 条边”的分层模型里也成立。也就是说，从概率模型投影到定边数模型，最优结构依旧保持不变。

难点（对应此处）：稳定性结论到底在控制什么？

remark. 它不只是控制“数值接近上界”，而是控制“结构接近极值构造”。也就是从“几乎最优”推出“几乎是某个固定三角形族”，这是比单纯上界强得多的陈述。

Proof Idea for Corollary 1.5 (Original). The idea of the proof is to study the family of all graphs containing a graph from $\mathcal{F}$, and to apply Theorem 1.4 to it, together with some Chernoff-type concentration of measure results.

remark. 证明策略可以理解为三步：先做上闭包把定边数族嵌入到整体空间；再调用主定理得到测度上界；最后用集中不等式把概率结论拉回固定边数层，完成桥接。

拓展至超图。

Definition 1.3 (Definition 1.6 (Original))**.** *We say that a family $\mathcal{F}$ of hypergraphs on $[n]$ is odd-linear-dependency-intersecting if for any $G, H \in \mathcal{F}$ there exist $l \in \mathbb{N}$ and nonempty sets $A_1, \ldots, A_{2l+1} \in G \cap H$ such that*

$$A_1 \triangle \cdots \triangle A_{2l+1} = \emptyset.$$

remark. 这一定义把图论语言抽象成代数语言：要求交集中存在“奇数个元素异或为零”的依赖关系。这样就能把三角形问题嵌入更一般的 $\mathbb{Z}_2$ 线性依赖框架。

Definition 1.4 (Definition 1.7 (Original))**.** *A family $\mathcal{F}$ of subsets of $\mathbb{Z}_2^n$ is odd-linear-dependency-intersecting if for any two subsets $S, T \in \mathcal{F}$, there exist $l \in \mathbb{N}$ and non-zero vectors $v_1, \ldots, v_{2l+1} \in S \cap T$ such that*

$$v_1 + \cdots + v_{2l+1} = 0.$$

remark. 向量版本让“对称差”直接变成“模 2 加法”，因此后续 *Fourier* 角色函数和谱算子都能自然落在群结构上。这是从组合叙述过渡到分析工具的关键一步。

Odd-LD-Agreeing And Schur Terminology (Original). Naturally, an odd-linear-dependency-agreeing family is defined by replacing $G \cap H$ with $G \triangle H$, and replacing $S \cap T$ with $S \triangle T$. Note that a Schur triple is a linearly dependent set of size 3, so Schur-triple-intersecting implies odd-linear-dependency-intersecting. A family $\mathcal{F}$ of subsets of $\mathbb{Z}_2^n$ is a Schur-umvirate if there exists a non-zero Schur triple $\{x, y, x+y\}$ such that $\mathcal{F}$ consists of all subsets containing $\{x, y, x+y\}$. A family $\mathcal{F}$ is a Schur junta if there exists a Schur triple $\{x, y, x+y\}$ such that $\mathcal{F}$ consists of all subsets with prescribed intersection with $\{x, y, x+y\}$.

remark. 这几句把“*Schur-umvirate / Schur-junta*”和前面的 *triangle* 版本完全对齐：一个是“必须全含”，一个是“交集模式固定”。后文所有等号结构结论都围绕这两个模板。

Theorem 1.4 (Theorem 1.8 (Original))**.** *Let $p \leq 1/2$, and let $\mathcal{F}$ be an odd-linear-dependency-intersecting family of subsets of $\mathbb{Z}_2^n$. Then*

$$\mu_p(\mathcal{F}) \leq p^3.$$

Equality holds if and only if $\mathcal{F}$ is a Schur-umvirate. Moreover, for each $p \leq 1/2$ there exists a constant c_p (bounded for $p \in [\delta, 1/2]$, for any fixed $\delta > 0$) such that for any $\varepsilon > 0$, if $\mu_p(\mathcal{F}) \geq p^3 - \varepsilon$ then there exists a Schur-umvirate $\mathcal{T}$ such that

$$\mu_p(\mathcal{T} \triangle \mathcal{F}) \leq c_p \varepsilon.$$

For $p = 1/2$, the corresponding statements hold for odd-linear-dependency-agreeing families.

remark. 这说明主现象并不依赖“图”本身，而依赖“奇线性依赖”这一骨架。换句话说，*triangle* 是一个具体外观，*odd dependency* 才是底层机制。

Post-Theorem Remarks (Original). This is indeed a generalization: any triangle-intersecting family of graphs can be lifted to a Schur-triple-intersecting family in the natural way. Also, $\{0,1\}^n$ can be viewed as a vector matroid over $\mathbb{Z}_2$, and any odd linear dependency contains a minimal odd dependency (an odd circuit), so Theorem 1.8 can also be interpreted as an odd-circuit-intersecting theorem in a $\mathbb{Z}_2$-matroid. The proof of Theorem 1.8 is deferred to Section 5.

remark. 这段在方法上很关键：作者强调“图论问题只是外层壳”，真正稳定的结构来自 $\mathbb{Z}_2$ 线性依赖与 *matroid* 奇回路。

Original Follow-up Remarks. Original follow-up remarks:

- this is a genuine generalization: triangle-intersecting graph families can be lifted to Schur-triple-intersecting hypergraph families;
- in $\{0,1\}^n$ (viewed as a vector matroid over $\mathbb{Z}_2$), odd linear dependencies contain minimal odd dependencies (odd circuits);

- hence Theorem 1.8 may also be viewed as an odd-circuit-intersecting theorem in a $\mathbb{Z}_2$-matroid setting.

remark. 作者在这里强调的是“抽象层级提升”：从图上的三角形，升维到 *vector matroid*（由矩阵列向量的线性无关关系定义出的拟阵）的奇回路约束，现象依旧稳定。

Methodological Observation (Original). *The ground set is a vector space over $\mathbb{Z}_2$; a triangle is not only a “triangle”, but an odd linear dependency over $\mathbb{Z}_2$. This makes discrete Fourier analysis a natural choice.*

remark. 这一句是方法论核心。只要对象本质上受“奇偶/异或”控制，*Fourier* 分析就会比纯计数工具更直接，因为它把“依赖”转成频谱上的可估量量。

1.3 Historical Background

Prior Bound In [4] (Original). *In [4], if $\mathcal{F}$ is triangle-intersecting, then $\mu(\mathcal{F}) \leq 1/4$; this improves the trivial bound $1/2$.*

remark. 历史上先达到 1/4，本文进一步压到 *sharp* 的 1/8（均匀测度）。这体现了方法升级带来的精度提升。

Original Remarks Related To [4]. Original remarks related to [4]:

- because the argument there uses bipartiteness, it already extends from triangle-intersecting to odd-cycle-intersecting;
- for $p = 1/2$, odd-cycle-agreeing can be shifted to odd-cycle-intersecting with the same measure;
- the basic observation can be written as $G \in \mathcal{F}$, B bipartite $\Rightarrow G \oplus B \notin \mathcal{F}$;
- for $p \leq 1/2$, [4] yields a p^2-type bound, while this paper improves to p^3 and conjectures validity up to $p \leq 3/4$.

remark. 这一组历史注记解释了本文创新点：不是从零开始，而是在既有熵方法基础上换到更强的谱路径。

Connection To [8] (Original). The paper [8] on t-intersecting families under product measure is a key thematic predecessor. For $S \subset [n]$,

$$\mu_p(S) = p^{|S|}(1-p)^{n-|S|}, \qquad \mu_p(\mathcal{F}) = \sum_{S \in \mathcal{F}} \mu_p(S).$$

For $p < \frac{1}{t+1}$, the unique largest-measure t-intersecting families are t-umvirates.

remark. 作者在这里等于是在说：当前论文的谱构造与稳定性套路，是沿着 *[8]* 的技术轨道向图论问题迁移而来。

Classical t-Intersecting Context (Original). For fixed $t \geq 2$, a family of subsets of $[n]$ is t-intersecting if all pairwise intersections have size at least t. The Erdős–Ko–Rado theorem gives the extremal model in large-n regimes, while the Ahlswede–Khachatrian complete intersection theorem characterizes the largest t-intersecting families for all n, k, t. Under product measure, for $p < 1/(t+1)$, [8] shows that the unique largest-measure t-intersecting families are t-umvirates, together with stability.

remark. 这段是“祖先问题”定位：本文不是孤立技巧，而是把成熟的 *t-intersecting* 谱框架迁移到了图空间与奇环约束。

Core Observation (Original). *Given triangle-intersecting $\mathcal{F}$ and bipartite B, for any $F_1, F_2 \in \mathcal{F}$, the set $F_1 \cap F_2$ must intersect B non-trivially.*

remark. 这是许多后续算子的出发点：二分图不含奇环，因此可以作为“排除模板”去制造强约束。

Shift Equivalence (Original). *A triangle-agreeing family can be transformed, via monotone shifts, into a triangle-intersecting family of the same size.*

remark. 这条等价桥梁很有用：某些情形下 *agreeing* 更好算，算完可借 *shift* 机制转回 *intersecting* 结论。

1.4 Method Framework

Operator Motivation (Original). *If $\mathcal{F}$ is triangle-intersecting (or odd-cycle-agreeing) and B is bipartite, then*

$$G \in \mathcal{F} \;\Rightarrow\; G \oplus B \notin \mathcal{F}.$$

So flipping edges of B moves graphs out of the family.

remark. 这一步把组合条件变成了运算闭包失效条件，进而可以转写为算子恒等式；这是从“看图”到“看线性变换”的关键转场。

Operator Lift (Original). *Let us lift this operation to an operator A_B on functions over subgraphs of K_n (equivalently $\mathbb{Z}_2^{\binom{n}{2}}$):*

$$A_B f(G) = f(G + B).$$

If B is random from distribution $\mathcal{B}$, define

$$A_{\mathcal{B}} f(G) = \mathbb{E}[f(G + B)].$$

remark. 这里把“翻边”动作算子化后，可以系统研究特征值与特征函数。很多极值问题在这个步骤之后就转化成谱半径或最小特征值估计问题。

Key Identity (Original). *If $f = 1_{\mathcal{F}}$, then $f(G) = 1 \Rightarrow A_{\mathcal{B}} f(G) = 0$, hence*

$$\langle f, A_{\mathcal{B}} f \rangle = 0.$$

Linear combinations of such operators preserve this identity.

remark. 这是全文最关键的代数约束之一。后续只要把 f 展开到 *Fourier* 基，结合算子特征值，就能把“零内积”翻译成关于频谱质量分布的不等式，从而推导上界与稳定性。

Linear Combination Step (Original). For coefficients c_B (not necessarily positive), define

$$A = \sum_B c_B A_B.$$

Then the key identity is preserved:

$$\langle f, Af \rangle = 0. \tag{1}$$

The next step is to identify eigenvalues/eigenfunctions of A and feed (1) into the Fourier side of f.

remark. 这就是全文的“谱引擎”：把组合约束压缩成一个二次型等式，再用特征值把不同频段逐个控制。

难点（对应此处）：为什么 $\langle f, Af \rangle = 0$ 这么关键？

remark. 因为它是“可全局分解”的约束。一旦投影到特征基上，它会变成一串可比较的权重关系，从而直接产出 *sharp* 上界和稳定性误差界。

Next Technical Step (Original). *The next step is to identify eigenvalues/eigenfunctions of A, then use the identity above to control the Fourier transform of f, and finally deduce structure of $\mathcal{F}$.*

remark. 整条证明链可以记成：构造算子 $\Rightarrow$ 计算谱 $\Rightarrow$ 限制 *Fourier* 支持 $\Rightarrow$ 读出极值构造与近极值构造。

Method Difficulty Remark (Original). *Even after finding the spectral route, the hard part is choosing the distributions $\mathcal{B}$ and coefficients c_B so that the resulting eigenvalues are exactly the right ones.*

remark. 这是第一部分里最“工程化”的难点：不是知道要用谱就够了，而是要反向设计出一组分布与权重，让最小特征值正好卡在目标值并保留谱间隙。

1.5 Structure Of The Paper

Roadmap (Original). *The paper treats $p = 1/2$ and $p < 1/2$ separately; then extends to Schur-triple-intersecting families, and ends with open problems.*

remark. 阅读时可以把它看成三层推进：均匀模型（最直观）$\to$ 偏置模型（技术更细）$\to$ 抽象推广（结构本质）。

Section 1 Supplement. *Section 1 sets the target values 1/8 (uniform measure) and p^3 (skew measure), and the rest of the paper constructs explicit spectra that force these bounds together with uniqueness and stability.*

remark. 补充一句主线：第一部分给的是“目标答案”，后续各节做的是“如何构造算子与谱，严格推出这个答案”。

2 Section 2 Notes

2.1 Fourier Analysis

Fourier Basics On $\mathbb{Z}_2^X$ (Original). We briefly recall the essentials of Fourier analysis on the Abelian group $\mathbb{Z}_2^X$, where X is finite. For any two functions $f, g : \mathbb{Z}_2^X \to \mathbb{R}$, define

$$\langle f, g\rangle = \mathbb{E}(fg) = \frac{1}{2^{|X|}} \sum_{S\subseteq X} f(S)g(S).$$

For each $R \subseteq X$, define

$$\chi_R(S) = (-1)^{|R\cap S|}.$$

Then $\chi_R(S \oplus T) = \chi_R(S)\chi_R(T)$, and $\{\chi_R : R \subseteq X\}$ is a complete orthonormal basis (the Fourier–Walsh basis). Hence every f has expansion

$$f = \sum_{R\subseteq X} \widehat{f}(R)\chi_R, \tag{2}$$

with $\widehat{f}(R) = \langle f, \chi_R\rangle$, and Parseval:

$$\langle f, g\rangle = \sum_{R\subseteq X} \widehat{f}(R)\widehat{g}(R).$$

If f is Boolean, then

$$\widehat{f}(\emptyset) = \mathbb{E}[f] = \mathbb{E}[f^2] = \sum_{R\subseteq X} \widehat{f}(R)^2.$$

For a family $\mathcal{F}$ and its indicator, this is

$$\mu(\mathcal{F}) = \widehat{\mathcal{F}}(\emptyset) = \sum_{R\subseteq X} \widehat{\mathcal{F}}(R)^2.$$

A useful formula is the convolution identity

$$\widehat{f * g} = \widehat{f} \cdot \widehat{g}, \qquad f * g(S) = \sum_{T\subseteq X} f(T)g(S+T).$$

remark. 这一小节是后续全部谱估计的坐标系：把家族问题转成 *Fourier* 系数的二次型问题，才能和 *OCC* 算子直接耦合。

难点（对应此处）：为什么 Boolean 情形下 $\mu(\mathcal{F}) = \sum \widehat{\mathcal{F}}(R)^2$ 这么重要？

remark. 因为它把“家族大小”变成“频谱总能量”。后面定理就是用算子把这份能量强行压在低阶（或 *tight* 图对应频段）上，从而导出上界和结构唯一性。

Key Mechanism (Original): energy, not counting. In the Fourier view, measure is identified with spectral energy, and operator constraints become algebraic identities on Fourier coefficients.

remark. 这一句是 *Section 2* 的底层逻辑：我们不再直接数家族元素，而是控制“能量如何分布在频谱上”。

2.2 Cayley Operators And Their Spectra

Spectral Route (Original). Largest intersecting-family questions can be converted into independent-set questions in suitable Cayley graphs, then controlled via Hoffman-type eigenvalue bounds and weighted perturbations.

remark. 这段是方法定位：先图论化（独立集），再谱化（特征值），最后通过加权构造把上界压到 *sharp* 值。*Hoffman* 方法 + 加权扰动

Definition 2.1 (Definition 2.1 (Odd-Cycle-Cayley Operator, Original)). *A linear operator A on real-valued functions on $\mathbb{Z}_2^{[n]^{(2)}}$ is called Odd-Cycle-Cayley (OCC) if:*

1. *for any odd-cycle-agreeing family $\mathcal{F}$ with indicator f,*
$$f(G) = 1 \Rightarrow Af(G) = 0;$$
2. *the Fourier–Walsh basis is a complete eigenbasis of A.*

remark. 定义 *2.1* 把组合约束和可对角化要求合并成一个对象。没有第 *1* 条就无法编码 *odd-cycle* 条件，没有第 *2* 条就无法做频谱计算。

Spectrum Notation (Original). For each $G \subseteq K_n$, let λ_G be the eigenvalue of χ_G. Write $\Lambda = (\lambda_G)_{G \subseteq K_n}$ for the OCC spectrum, $\lambda_{\min}$ for its minimum, and
$$\Lambda_{\min} = \{G : \lambda_G = \lambda_{\min}\}.$$
The spectral gap is the maximal γ such that
$$\lambda_H \geq \lambda_{\min} + \gamma, \qquad H \notin \Lambda_{\min}.$$
OCC operators (hence OCC spectra) form a linear space.

remark. 线性空间性质是后面“先做一套主谱，再加一套修正谱”的根本原因。

Operator Construction (Original). Let B be a bipartite graph, and define
$$A_B f(G) = f(G + B).$$
If $\mathcal{B}$ is a distribution over bipartite graphs, define
$$A_{\mathcal{B}} f(G) = \mathbb{E}[f(G \oplus B)].$$

remark. 这是 *Section 1* 里“翻边”动作的算子化版本。

Markov Walk View (Original). The same operator can be viewed as a random walk on graph space with stationary uniform measure; if $\mathcal{F}$ is odd-cycle-intersecting, two consecutive states cannot both lie in $\mathcal{F}$.

remark. 这个视角在后面 $p < 1/2$ 时尤其重要，因为偏置测度下会继续沿着“随机过程 + 谱”这条路线推进。

Claim 2.1 (Claim 1 (Original))**.** *$A_{\mathcal{B}}$ is an OCC operator, and its spectrum is*

$$\lambda_R = (-1)^{|R|}\,\mathbb{E}[\chi_B(R)].$$

Proof. If $\mathcal{F}$ is odd-cycle-agreeing and $f = 1_{\mathcal{F}}$, then **key annihilation step**

$$f(G) = 1 \Rightarrow A_{\mathcal{B}} f(G) = 0.$$

Also, for Fourier characters: **key eigenfunction step**

$$A_{\mathcal{B}}\chi_R(G) = \mathbb{E}[\chi_R(G \oplus B)] = \chi_R(G)\,\mathbb{E}[\chi_R(B)],$$

so χ_R is an eigenfunction with eigenvalue

$$\lambda_R = \mathbb{E}[\chi_R(B)].$$

This can be rewritten in the claim form: **key spectral formula**

$$\lambda_R = (-1)^{|R|}\mathbb{E}[\chi_B(R)] = (-1)^{|R|}\mathbb{E}[\chi_B(R \cap B)]. \tag{3}$$

□

remark. 这条结论最有价值的地方是"把谱值表达成随机 *cut* 统计的期望"，从而把代数问题转成可算概率量。

Equivalent Views Of $A_{\mathcal{B}}$ (Original).

- It is a convolution operator, so Fourier–Walsh characters are eigenfunctions.
- It is an average of tensor-product coordinate operators.
- It is a Cayley-graph adjacency operator (on an Abelian group).
- It is a Markov operator for a random walk on graph space.

remark. 同一个算子的四种视角分别对应：分析、线性代数、图论、概率。后文在不同步骤会反复切换视角。

Theorem 2.2 (Theorem 2.2 (Original))**.** *Let $\Lambda = (\lambda_G)_{G \subseteq K_n}$ be an OCC spectrum with*

$$\lambda_\emptyset = 1, \qquad -1 < \lambda_{\min} < 0, \qquad \gamma > 0.$$

Set

$$\nu = -\frac{\lambda_{\min}}{1 - \lambda_{\min}} \quad \left(\text{equivalently } \lambda_{\min} = -\frac{\nu}{1-\nu}\right).$$

For any odd-cycle-agreeing family $\mathcal{F}$:

- *Upper bound: $\mu(\mathcal{F}) \le \nu$.*
- *Uniqueness: if $\mu(\mathcal{F}) = \nu$, then $\widehat{\mathcal{F}}(G) \neq 0$ only for $G \in \Lambda_{\min} \cup \{\emptyset\}$.*

- *Stability: letting*

$$w = \sum_{G \notin \Lambda_{\min} \cup \{\emptyset\}} \widehat{\mathcal{F}}(G)^2,$$

we have

$$w \leq \frac{\nu}{(1-\nu)\gamma}\,(\nu - \mu(\mathcal{F})).$$

Proof. Let A be an OCC operator with spectrum Λ. Then

$$A(\mathcal{F}) = \sum_G \lambda_G \widehat{\mathcal{F}}(G)\chi_G,$$

and thus **key quadratic identity**

$$0 = \langle \mathcal{F}, A\mathcal{F} \rangle = \sum_G \lambda_G \widehat{\mathcal{F}}(G)^2.$$

Since $\widehat{\mathcal{F}}(\emptyset) = \sum_G \widehat{\mathcal{F}}(G)^2 = \mu(\mathcal{F})$ and

$$w = \sum_{G \notin \Lambda_{\min} \cup \{\emptyset\}} \widehat{\mathcal{F}}(G)^2,$$

we get **key mass decomposition**

$$\sum_{G \in \Lambda_{\min}} \widehat{\mathcal{F}}(G)^2 = \mu(\mathcal{F}) - \mu(\mathcal{F})^2 - w.$$

Therefore, **core spectral inequality**

$$\begin{aligned} 0 &= \sum_G \lambda_G \widehat{\mathcal{F}}(G)^2 \\ &\geq \lambda_\emptyset \mu(\mathcal{F})^2 + \lambda_{\min}\big(\mu(\mathcal{F}) - \mu(\mathcal{F})^2 - w\big) + (\lambda_{\min} + \gamma)w \\ &= \mu(\mathcal{F})^2 - \frac{\nu}{1-\nu}\big(\mu(\mathcal{F}) - \mu(\mathcal{F})^2 - w\big) + w\gamma \\ &= \frac{\mu(\mathcal{F})^2}{1-\nu} - \frac{\mu(\mathcal{F})\nu}{1-\nu} + w\gamma. \end{aligned}$$

Hence **key bound transfer**

$$\mu(\mathcal{F})^2 - \mu(\mathcal{F})\nu + w\gamma(1-\nu) \leq 0.$$

Since $\gamma > 0$, this implies $\mu(\mathcal{F}) \leq \nu$, with equality iff $w = 0$. Also,

$$\mu(\mathcal{F})\big(\mu(\mathcal{F}) - \nu\big) + w\gamma(1-\nu) \leq 0,$$

so **key stability estimate**

$$w \leq \frac{\mu(\mathcal{F})}{(1-\nu)\gamma}\,(\nu - \mu(\mathcal{F})) \leq \frac{\nu}{(1-\nu)\gamma}\,(\nu - \mu(\mathcal{F})).$$

□

remark. Theorem 2.2 是 *Section 2* 的总模板：同一个二次型不等式同时产出上界、等号频谱支撑和稳定性误差界。

Key Quadratic Constraint (Original). The whole theorem is driven by the identity

$$\langle \mathcal{F}, A\mathcal{F}\rangle = 0,$$

which turns combinatorial restrictions into a spectral quadratic inequality.

remark. 这条等式是"组合条件转代数约束"的桥，后面的所有 *sharp* 常数都由它展开。

Corollary 2.3 (Corollary 2.3 (Original)). *Suppose there exists an OCC spectrum with*

$$\lambda_\emptyset = 1, \qquad \lambda_{\min} = -\frac{1}{7}, \qquad \gamma > 0,$$

and all graphs in $\Lambda_{\min}$ have at most three edges. Then for any odd-cycle-agreeing family $\mathcal{F} \subseteq 2^{E(K_n)}$:

- $\mu(\mathcal{F}) \leq \frac{1}{8}$;
- *if $\mu(\mathcal{F}) = \frac{1}{8}$, then $\mathcal{F}$ is a triangle junta;*
- *if $\mu(\mathcal{F}) \geq \frac{1}{8} - \varepsilon$, then there exists a triangle junta $\mathcal{T}$ such that*

$$\mu(\mathcal{F} \triangle \mathcal{T}) \leq c\varepsilon$$

for an absolute constant $c > 0$.

Proof. **Upper bound key step:** The upper bound follows directly from Theorem 2.2.

Uniqueness. First assume $\mathcal{F}$ is odd-cycle-intersecting; the reduction from agreeing to intersecting is Lemma 2.7 below. **key low-degree reduction:** since the Fourier transform is supported on graphs with at most three edges, Nisan–Szegedy implies $\mathcal{F}$ depends on at most $3 \cdot 2^3 = 24$ coordinates. We may additionally assume $\mathcal{F}$ is an up-set (replace by up-filter if needed), and $\mu(\mathcal{F}) = 1/8$. Then Lemma 2.4 applies: **$\mathcal{F}$ is a 3-umvirate**; in this setting, triangle-intersecting forces it to be a $\triangle$umvirate. Finally, Lemma 2.7 gives the odd-cycle-agreeing-to-junta reduction.

Stability. Use Theorem 2.5 (Kindler–Safra). Assume $\mu(\mathcal{F}) > 1/8 - \varepsilon$. By Theorem 2.2, **key Fourier-tail estimate**

$$w = \sum_{G \notin \Lambda_{\min} \cup \{\emptyset\}} \widehat{\mathcal{F}}(G)^2 \leq \frac{1}{7\gamma}\varepsilon.$$

Applying Kindler–Safra with $t = 3$, provided $\varepsilon \leq 7\gamma\varepsilon_0$, **$\mathcal{F}$ is quantitatively close** to some family depending on boundedly many coordinates:

$$\left(\frac{c_0}{7\gamma}\varepsilon\right)\text{-close}$$

to some family depending on at most T_0 coordinates. There are only finitely many such families that are not triangle juntas; by uniqueness, all of them have measure $< 1/8$. Choose $\varepsilon_1 > 0$ so that each has measure $< 1/8 - \left(1 + \frac{c_0}{7\gamma}\right)\varepsilon_1$. If $\varepsilon \leq \varepsilon_1$, $\mathcal{F}$ cannot be that close to any of them while still having measure at least $1/8 - \varepsilon$. Hence **the approximating family must be a triangle junta**. If $\varepsilon > \min(7\gamma\varepsilon_0, \varepsilon_1) =: \varepsilon_2$, take $c = 1/\varepsilon_2$. $\square$

remark. 这个证明的结构是"谱稳定性 → 有限坐标逼近 → 排除有限坏例"：它不是直接分类全部族，而是借有限性做反证收口。

Key Reduction In Uniqueness (Original). The uniqueness proof combines three ingredients in order: low-degree Fourier support ⇒ bounded-coordinate dependence ⇒ umvirate/junta rigidity.

remark. 这三个步骤缺一不可：只有低阶支持不够，必须再借单调性与结构引理把“低维”压成“固定三角形”。

Lemma 2.4 (Lemma 2.4 ([8]). , *Original)] Let $N \in \mathbb{N}$, let $p \le 1/2$, and let $f : \{0,1\}^N \to \{0,1\}$ be monotone with*

$$\mathbb{E}_p f = p^t, \qquad \widehat{f}(S) = 0 \text{ for all } |S| > t.$$

Then f is a t-umvirate (depends only on t coordinates).

remark. 这条是“低阶频谱集中 + 单调性 ⇒ 纯 umvirate 结构”的关键黑箱。

Theorem 2.5 (Theorem 2.5 (Kindler–Safra, Original)). *For every $t \in \mathbb{N}$, there exist $\varepsilon_0 > 0$, $c_0 > 0$, $T_0 \in \mathbb{N}$ such that: if $f : \{0,1\}^N \to \{0,1\}$ satisfies*

$$\sum_{|S|>t} \widehat{f}(S)^2 = \varepsilon < \varepsilon_0,$$

then there exists Boolean $g : \{0,1\}^N \to \{0,1\}$ depending on at most T_0 coordinates with

$$\mu\big(\{R : f(R) \ne g(R)\}\big) \le c_0\varepsilon.$$

remark. 这里最强的一点是：只要高阶尾质量足够小，逼近函数所需坐标数就被统一上界住，不随 ε 继续恶化。

难点（对应此处）：为什么 Corollary 2.3 的稳定性要引入“有限坏族排除”？

remark. 因为 *Kindler–Safra* 先给的是“接近某个低维 *Boolean* 族”，但这个族未必就是 *triangle junta*。要得到最终结构，必须用唯一性结果把非 *junta* 的有限候选全部排掉。

2.3 The Intersecting/Agreeing Equivalence

Abstract Setup (Original). Let X be finite and $\mathcal{Z} \subseteq P(X)$. A family $\mathcal{A} \subseteq P(X)$ is $\mathcal{Z}$-intersecting if for all $A, B \in \mathcal{A}$ there exists $Z \in \mathcal{Z}$ with $Z \subseteq A \cap B$. It is $\mathcal{Z}$-agreeing if for all $A, B \in \mathcal{A}$ there exists $Z \in \mathcal{Z}$ with

$$Z \cap (A \triangle B) = \emptyset.$$

Define

$$m(\mathcal{Z}) = \max\{|\mathcal{A}| : \mathcal{A} \subseteq P(X),\ \mathcal{A} \text{ is } \mathcal{Z}\text{-intersecting}\},$$
$$\bar{m}(\mathcal{Z}) = \max\{|\mathcal{A}| : \mathcal{A} \subseteq P(X),\ \mathcal{A} \text{ is } \mathcal{Z}\text{-agreeing}\}.$$

remark. 这一步把图论中的 *agreeing/intersecting* 问题抽象成集合系统命题，方便直接调用压缩（*monotonization*）方法。

Monotonization Principle (Original). Compression operators C_i preserve family size and the agreeing property, while driving the family toward an up-set; once up-closed, agreeing upgrades to intersecting.

remark. 这是 *Lemma 2.6* 的灵魂：操作本身不损规模，却提升结构可控性，最终把 *agreeing* 问题化成 *intersecting* 问题。

Lemma 2.6 (Lemma 2.6 (Original)). *For finite X and $\mathcal{Z} \subseteq P(X)$,*

$$m(\mathcal{Z}) = \bar{m}(\mathcal{Z}).$$

Proof. **first inequality** Clearly, every $\mathcal{Z}$-intersecting family is $\mathcal{Z}$-agreeing, so

$$m(\mathcal{Z}) \leq \bar{m}(\mathcal{Z}).$$

It remains to show that any $\mathcal{Z}$-agreeing family can be transformed into a $\mathcal{Z}$-intersecting family of the same size.

For $i \in X$, define the i-monotonization C_i: for family $\mathcal{A} \subseteq P(X)$, replace each A by $A \cup \{i\}$ whenever

$$i \notin A, \quad A \in \mathcal{A}, \quad A \cup \{i\} \notin \mathcal{A}.$$

Then $|C_i(\mathcal{A})| = |\mathcal{A}|$, and if $\mathcal{A}$ is $\mathcal{Z}$-agreeing then so is $C_i(\mathcal{A})$.

Starting from $\mathcal{F}_0 = \mathcal{F}$, repeatedly apply such C_i whenever possible: if there exist $F \in \mathcal{F}_k$ and $i \in X$ with $F \cup \{i\} \notin \mathcal{F}_k$, set

$$\mathcal{F}_{k+1} = C_i(\mathcal{F}_k).$$

At each step, the sum of set sizes in the family increases by at least 1, so the process terminates at some $\mathcal{F}_\ell$. Let $\mathcal{F}' = \mathcal{F}_\ell$. Then $\mathcal{F}'$ is $\mathcal{Z}$-agreeing, $|\mathcal{F}'| = |\mathcal{F}|$, and $\mathcal{F}'$ is an **up-set**.

Now for $F, G \in \mathcal{F}'$, we have $F \cup G \in \mathcal{F}'$. Since $\mathcal{F}'$ is $\mathcal{Z}$-agreeing, there exists $Z \in \mathcal{Z}$ such that

$$((F \cup G) \triangle G) \cap Z = \emptyset.$$

But $(F \cup G) \triangle G = F \setminus G$, so this implies $Z \subseteq F \cap G$. Hence **$\mathcal{F}'$ is $\mathcal{Z}$-intersecting**, therefore

$$\bar{m}(\mathcal{Z}) \leq m(\mathcal{Z}).$$

So $m(\mathcal{Z}) = \bar{m}(\mathcal{Z})$. □

remark. *Lemma 2.6* 的核心是"压缩后上闭 + *agreeing* 条件自动升级成 *intersecting* 条件"。

Lemma 2.7 (Lemma 2.7 (Original)). *Let $\mathcal{F}$ be odd-cycle-agreeing. Assume a sequence of monotonizations C_e ($e \in [n]^{(2)}$) produces a family $\mathcal{F}_k$ that is a $\triangle$umvirate. Then $\mathcal{F}$ is a triangle junta.*

Proof. Suppose $\mathcal{F}_k$ is odd-cycle-agreeing and $\mathcal{F}_{k+1} = C_e(\mathcal{F}_k) \neq \mathcal{F}_k$ is a T-junta for some triangle $T \subseteq K_n$. Then there exists $G \notin \mathcal{F}_{k+1}$ with $G \cup \{e\} \in \mathcal{F}_{k+1}$; since $\mathcal{F}_{k+1}$ is a T-junta, necessarily $e \in T$. Let $S \subseteq T$ satisfy

$$\mathcal{F}_{k+1} = \{G \in \mathbb{Z}_2^{[n]^{(2)}} : G \cap T = S\}.$$

Clearly $e \in S$.

Define

$$\mathcal{C} = \{G \in \mathcal{F}_k : e \notin G\}, \qquad \mathcal{D} = \{G \in \mathcal{F}_k : e \in G\},$$

so $\mathcal{F}_k = \mathcal{C} \sqcup \mathcal{D}$. If $G \in \mathcal{C}$, then $G \cup \{e\} \notin \mathcal{D}$ (else both are in $\mathcal{F}_{k+1}$, contradicting fixed $G \cap T = S$ in the junta). Hence

$$\mathcal{F}_{k+1} = \mathcal{D} \sqcup \{G \cup \{e\} : G \in \mathcal{C}\}.$$

Therefore every $G \in \mathcal{D}$ has $G \cap T = S$, and every $G \in \mathcal{C}$ has $G \cap T = S - e$.

Since $\mathcal{F}_{k+1} \neq \mathcal{F}_k$, we have $\mathcal{C} \neq \emptyset$. **key claim:** we claim $\mathcal{D} = \emptyset$. Assume not; let $|\mathcal{D}| = N \geq 1$, then

$$|\mathcal{C}| = 2^{\binom{n}{2}-3} - N \geq 1.$$

Because $T \oplus e$ intersects every triangle, if $G \in \mathcal{F}_k$ then

$$G \oplus (T \oplus e) \notin \mathcal{F}_k.$$

Since $|\mathcal{C}| + |\mathcal{D}| = 2^{\binom{n}{2}-3}$, for every $H \subseteq T$ exactly one of

$$H \oplus S \in \mathcal{D}, \qquad H \oplus S \oplus T \oplus e \in \mathcal{C}$$

holds. So

$$V = \{H \subseteq T : H \oplus S \in \mathcal{D}\}, \quad W = \{H \subseteq T : H \oplus S \oplus T \oplus e \in \mathcal{C}\}$$

partition labelled subgraphs of T, with both nonempty. Hence two adjacent subgraphs lie in different classes: there exist $H \subseteq T$ and $f \in E(T)$ such that

$$H \oplus S \in \mathcal{D}, \qquad H \oplus f \oplus S \oplus T \oplus e \in \mathcal{C}.$$

These two graphs agree only on $T \oplus e \oplus f$, a 3-edge graph containing exactly two edges of T, thus not a triangle, **contradicting odd-cycle-agreeing** of $\mathcal{F}_k$.

Therefore $\mathcal{D} = \emptyset$, i.e.

$$\mathcal{F}_k = \{G \in \mathcal{G}_n : G \cap T = S - e\},$$

so $\mathcal{F}_k$ is also a T-junta. **backward induction closes the proof:** $\mathcal{F}_0 = \mathcal{F}$ is a T-junta. □

remark. *Lemma 2.7* 的技术点是：把一次压缩前后的差异编码成 $\mathcal{C}, \mathcal{D}$，再用"相邻子图"反证 *agreeing* 条件，从而把 *junta* 结构向前逐步回传。

难点（对应此处）：为什么要引入 V, W 这两个类？

remark. 它们把"压缩前后互补映射"离散成一个二分类问题。只要两类都非空，就必有一条邻边跨类，跨类点正好给出违反 *agreeing* 的一对图。

2.4 Constructing The Required OCC Spectrum

Two-Step Plan (Original). First construct an OCC spectrum $\Lambda^{(1)}$ with correct minimum value but extra tight graphs (all 4-forests and K_4^-). Then add a multiple of $\Lambda^{(2)}$, which is positive on problematic graphs and zero on all graphs with at most three edges.

remark. 这是 *Section 2* 的核心工程：主谱定标，修正谱清噪。

Lemma 2.8 (Lemma 2.8 (Original)). *Let $\mathcal{B}$ be a distribution on bipartite graphs, and for each $B \in \mathcal{B}$ let f_B be a real-valued function on subgraphs of B. Then*

$$\lambda_G = (-1)^{|G|}\,\mathbb{E}[f_B(B \cap G)]$$

is an OCC spectrum.

Proof. Fix bipartite B. By Claim 1, A_B is OCC; equivalently,

$$\lambda_G = (-1)^{|G|}\chi_{B'}(G \cap B)$$

is an OCC spectrum for each $B' \subseteq B$. Since $\{\chi_{B'} : B' \subseteq B\}$ **spans all functions** on subgraphs of B, for any f the vector

$$\lambda_G = (-1)^{|G|} f(G \cap B)$$

is OCC. **taking expectation** over random $B \sim \mathcal{B}$ gives the statement. □

remark. Lemma 2.8 说明：只要你能把目标特征值写成 cut 子图统计的线性组合，就自动得到一个 OCC 谱。

Concrete Choice Of Building Blocks (Original). The distribution $\mathcal{B}$ is the uniform distribution on complete bipartite subgraphs from random bipartitions. Functions f_B are chosen as indicators of:

- cut size events $|G \cap B| = i$ $(i = 1, 2, 3)$,
- or cut-isomorphism events $G \cap B \cong R$ for specific bipartite R.

remark. 这就是为什么后面会出现 $q_i(G)$ 和 $q_R(G)$：它们正是可控且可组合的“谱基底坐标”。

Engineering Insight (Original). The $q_i(G)$ and $q_R(G)$ coordinates are rich enough to interpolate exact values on small graphs, while decaying fast enough on large graphs to preserve a strict spectral gap.

remark. 这句话解释了为何构造不是随便拼系数：它要同时满足“小图精确命中”和“大图统一压开”两个目标。

Corollary 2.9 (Corollary 2.9 (Original)). *Let (V_1, V_2) be a random bipartition of $V(K_n)$ (each vertex independently chooses side with probability $1/2$), and let B be the cut edges between V_1, V_2. For $G \subseteq K_n$, define*

$$q_i(G) = \Pr(|G \cap B| = i),$$

and for bipartite R,

$$q_R(G) = \Pr(G \cap B \cong R).$$

Then for every integer i,

$$\lambda_G = (-1)^{|G|} q_i(G)$$

is an OCC spectrum, and for every bipartite R,

$$\lambda_G = (-1)^{|G|} q_R(G)$$

is also an OCC spectrum.

remark. Corollary 2.9 把抽象模板直接落成可计算对象，后文所有显式系数都在这里起源。

Claim 2.10 (Claim 2 (Original)). *Define*

$$\lambda_G^{(1)} = (-1)^{|G|}\left[q_0(G) - \frac{5}{7}q_1(G) - \frac{1}{7}q_2(G) + \frac{3}{28}q_3(G)\right].$$

Then:

- $\lambda_\emptyset^{(1)} = 1$;
- $\lambda_{\min}^{(1)} = -1/7$;
- $\Lambda_{\min}^{(1)}$ *consists of: a single edge, a path of length two, two disjoint edges, a triangle, all 4-forests, and* K_4^-;
- *for all* $H \notin \Lambda_{\min}^{(1)}$,

$$\lambda_H^{(1)} \geq -\frac{1}{7} + \gamma', \qquad \gamma' = \frac{1}{56}.$$

remark. Claim 2 给出主谱，但 tight 集里混入了 4-forest 与 K_4^-，所以还要第二步修正。

Claim 2.11 (Claim 3 (Original)). *Let*

$$\lambda_G^{(2)} = (-1)^{|G|}\left[\sum_{F \in \mathcal{F}_4} q_F(G) - q_{C_4}(G)\right],$$

where the sum is over all 4-forests. Then:

1. $\lambda_H^{(2)} = 0$ *for all H with fewer than four edges;*
2. $\lambda_F^{(2)} = 1/16$ *for all 4-forests F;*
3. $\lambda_{K_4^-}^{(2)} = 1/8$;
4. $|\lambda_G^{(2)}| \leq 1$ *for all G.*

Proof. 1. Clear: a cut in a graph with at most three edges has size at most 3.

2. For any forest, each edge belongs to a random cut independently; hence $q_F(F) = 2^{-|F|} = 1/16$ for $|F| = 4$. Also $q_{C_4}(F) = 0$, and $q_F(F') = 0$ for distinct 4-forests F, F'.

3. Label K_4^- vertices by a, b, c, d, where a, c have degree 3. A random cut is isomorphic to C_4 iff a, c are on one side and b, d on the other, **probability** $1/8$. Also $q_F(K_4^-) = 0$ for every 4-forest.

4. $|\lambda_G^{(2)}|$ is a difference of two probabilities, so it is at most 1.

$\square$

remark. Claim 3 的作用是"只动坏点，不动低阶目标点"：它对 $|G| \leq 3$ 全零，因此不会破坏主谱在关键小图上的定标。

Proof Placement (Original). The proof of Claim 2 is deferred to Section 3 (cut statistics analysis).

remark. 这也是 *Section 2* 与 *Section 3* 的接口：*Section 2* 负责"谱设计目标"，*Section 3* 负责"概率估计兑现"。

Corollary 2.12 (Corollary 2.10 (Original))**.** *Let*

$$\Lambda = \Lambda^{(1)} + \frac{16}{17}\gamma'\Lambda^{(2)}.$$

Then Λ is an OCC spectrum as required by Corollary 2.3:

- $\lambda_\emptyset = 1$*;*
- $\lambda_G = -1/7$ *for all non-empty subgraphs G of K_3 (and for the graph of two disjoint edges);*
- *with $\gamma = \gamma'/17$, for every G with more than three edges,*

$$\lambda_G \geq -\frac{1}{7} + \gamma.$$

Proof. For any 4-forest F, **lift bad tight points:**

$$\lambda_F = -\frac{1}{7} + \frac{16}{17}\gamma' \cdot \frac{1}{16} = -\frac{1}{7} + \frac{\gamma'}{17}.$$

For K_4^-,

$$\lambda_{K_4^-} = -\frac{1}{7} + \frac{16}{17}\gamma' \cdot \frac{1}{8} = -\frac{1}{7} + \frac{2\gamma'}{17}.$$

For all other non-empty graphs G,

$$\lambda_G \geq -\frac{1}{7} + \gamma' - \frac{16}{17}\gamma' = -\frac{1}{7} + \frac{\gamma'}{17}.$$

So **taking** $\gamma = \gamma'/17$ yields the desired gap. $\square$

remark. Corollary 2.10 完成了 $p = 1/2$ 的谱闭环：值、*tight* 集、*gap* 三个条件同时就位。

Section 2 Closing Point (Original). After Corollary 2.10, the uniform-measure case is reduced to verifying Claim 2, which is exactly the cut-statistics analysis in Section 3.

remark. 所以 *Section 2* 的成果是"证明框架 + 目标谱"，*Section 3* 则负责最后的概率估计落地。

3 Section 3 Notes

3.1 Cut Statistics

Section Goal (Original). The purpose of this section is to study cut statistics of graphs under a uniform random cut, in order to prove Claim 2 from Section 2.

remark. Section 2 给了谱构造目标，*Section 3* 负责把这些目标变成可验证的不等式与精确值。

Block-Decomposition Setup (Original). We begin by using block decompositions of graphs to simplify the calculations.

remark. 先分块再算分布，是这一节把复杂图问题降维的第一步。

Random-Cut Model (Original). We may view a random cut in G as a random red/blue colouring

$$c : V(G) \to \{\text{red}, \text{blue}\},$$

with each vertex independently red/blue with probability 1/2. Let $Y(c)$ be the number of multi-coloured edges (equivalently, cut edges).

remark. 这个视角让"*cut* 边数"成为随机变量，后续所有 q_k 都是在统计这个随机变量的分布。

Cut Distribution Notation (Original). Let

$$Q(G) = (q_k(G))_{k\geq 0}$$

be the distribution of $|G \cap B|$ (a random cut size), and let

$$Q_G(X) = \sum_{k\geq 0} q_k(G)X^k$$

be its probability-generating function. For a single edge,

$$q_0(-) = q_1(-) = \frac{1}{2}, \qquad Q_-(X) = \frac{1}{2} + \frac{1}{2}X.$$

If Y_1, Y_2 are independent, then

$$Q_{Y_1+Y_2}(X) = Q_{Y_1}(X)Q_{Y_2}(X).$$

remark. 概率母函数的乘法性质是这一节的主工具：一旦切分成独立块，分布计算就变成代数乘积。

Small-Graph Table Used To Engineer $\Lambda^{(1)}$ (Original). For key small graphs, the cut-distribution values are:

G	$q_0(G)$	$q_1(G)$	$q_2(G)$	$q_3(G)$	$q_4(G)$
$\emptyset$	1	0	0	0	0
$-$	1/2	1/2	0	0	0
$\wedge$	1/4	1/2	1/4	0	0
$\triangle$	1/4	0	3/4	0	0
F_4	1/16	4/16	6/16	4/16	1/16
K_4^-	1/8	0	1/4	1/2	1/8

where F_4 is any 4-edge forest.

remark. 这张表是系数反求的起点：先在小图上“打靶”，再去控制所有大图。

Coefficient Engineering For $\Lambda^{(1)}$ (Original). Suppose

$$\lambda(G) = (-1)^{|G|}\big[c_0q_0(G) + c_1q_1(G) + c_2q_2(G) + c_3q_3(G) + c_4q_4(G)\big].$$

From $\lambda(\emptyset) = 1$, we get $c_0 = 1$. Applying Theorem 2.2 to a $\triangle$umvirate forces

$$\lambda(G) = \lambda_{\min} = -\frac{1}{7}$$

for all subgraphs of a triangle, giving

$$c_1 = -\frac{5}{7}, \qquad c_2 = -\frac{1}{7}.$$

Substituting into the equations for F_4 and K_4^- yields

$$4c_3 + c_4 = \frac{3}{7}.$$

Choose $c_4 = 0$, hence

$$c_3 = \frac{3}{28}.$$

Both bounds coincide (what luck!) This good fortune does not hold for $p > 1/2$.

remark. 这一步说明 *Claim 2* 里的系数不是拍脑袋，而是由 *tight* 常数与小图约束反推出来的。

Key Engineering Step (Original): matching small graphs first. The spectrum $\Lambda^{(1)}$ is engineered to match the table exactly on the controlling small graphs; the rest of this section shows it also works for all other graphs.

remark. 先“小图精确命中”，再“大图统一估计”，这正是谱构造中最稳妥的路径。

Block Splitting And Independence (Original). If $G = G_1 \sqcup G_2$, then

$$Q_G(X) = Q_{G_1}(X)Q_{G_2}(X),$$

since $|G \cap B| = |G_1 \cap B| + |G_2 \cap B|$ with independence.

If G is connected and v is a cutvertex splitting $G - v$ into $G[S_1], \dots, G[S_N]$, let

$$H_i = G[S_i \cup \{v\}].$$

Then the random variables $\{|H_i \cap B|\}_{i=1}^N$ are independent and

$$|G \cap B| = \sum_{i=1}^{N} |H_i \cap B|, \qquad Q_G(X) = \prod_{i=1}^{N} Q_{H_i}(X).$$

If $H = \bigsqcup_i H_i$ is the split at v, then

$$Q_G(X) = Q_H(X).$$

remark. *“split 不改变 cut 分布”是后续全部分解计算的核心结构事实。*

Global Split Formula (Original). Let G_s be the graph obtained by fully splitting G into blocks. If G has m bridges and t_K blocks isomorphic to each biconnected graph K, then

$$Q_G(X) = Q_{G_s}(X) = \left(\frac{1}{2} + \frac{1}{2}X\right)^m \prod_{K \in \mathcal{K}} (Q_K(X))^{t_K} = \frac{1}{2^m}(1+X)^m \prod_{K \in \mathcal{K}} (Q_K(X))^{t_K},$$

where $\mathcal{K}$ is a set of isomorphism-class representatives of biconnected graphs.

remark. *这个公式把任意图的 cut 分布压缩成“桥 + 双连通块”两部分，后面估计都会从这一步展开。*

Split Example (Original). If a graph is formed by joining two triangles with one bridge, then

$$Q_G(X) = \left(\frac{1}{2} + \frac{1}{2}X\right) (Q_\triangle(X))^2 = \left(\frac{1}{2} + \frac{1}{2}X\right) \left(\frac{1}{4} + \frac{3}{4}X^2\right)^2.$$

remark. *这个例子直观展示了“桥贡献一个 $(1+X)/2$ 因子，双连通块各自乘上去”。*

Coefficient Expansion Formula (Original). Suppose G has exactly m bridges. Let H be the union of biconnected components of G_s, and write

$$Q_H(X) = \sum_{i \geq 0} a_i X^i.$$

Then $\sum_{i \geq 0} a_i = 1$, and because H is bridgeless, $a_1 = 0$. Hence

$$\begin{aligned} Q_G(X) &= \left(\frac{1}{2} + \frac{1}{2}X\right)^m Q_H(X) \\ &= \frac{1}{2^m}(1+X)^m (a_0 + a_2X^2 + a_3X^3 + \cdots) \\ &= \frac{1}{2^m}\Big(a_0 + ma_0X + (\binom{m}{2}a_0 + a_2)X^2 \\ &\qquad + (\binom{m}{3}a_0 + ma_2 + a_3)X^3 + R(X)X^4\Big), \end{aligned} \tag{4}$$

for some $R(X) \in \mathbb{Q}[X]$.

remark. *式 (4) 是 Claim 2 证明的主计算公式：它把 q_0, q_1, q_2, q_3 都写成 m, a_0, a_2, a_3 的显式组合。*

3.2 Proof Of Claim 2

Lemma 3.1 (Lemma 3.1 (Original)). *Let G be a graph.*

1. *If G has exactly N connected components, then*

$$q_0(G) = 2^{N-v(G)}.$$

2. *If G has exactly m bridges, then*

$$q_1(G) = m\, q_0(G).$$

3. *If G has a vertex of odd degree, then $q_k(G) \le 1/2$ for all $k \ge 0$.*

4. *For odd k, $q_k(G) \le 1/2$.*

5. *Always $q_2(G) \le 3/4$.*

Proof.

1. $G \cap B = 0$ iff each connected component is monochromatic. This has probability $2^{N-v(G)}$.

2. This follows directly from (4).

3. Let v have odd degree. For any colouring c, flip the colour of v to get c'. Denote by $Y_v(c)$ (resp. $Y_v(c')$) the number of cut edges incident to v under c (resp. c'). Then

$$Y_v(c) + Y_v(c') = \deg(v),$$

so $Y_v(c) \ne Y_v(c')$ because $\deg(v)$ is odd. Also $Y(c) - Y_v(c) = Y(c') - Y_v(c')$, hence

$$Y(c) \ne Y(c').$$

Since $c \leftrightarrow c'$ is an involution, in each pair (c, c') at most one colouring has cut size k.

4. By item 3, we may assume all degrees are even. Then each connected component is Eulerian, so every cut has even size; hence for odd k, $q_k(G) = 0$.

5. The random-cut average size is $|G|/2$, so

$$\frac{|G|}{2} = \sum_k k q_k(G) < 2q_2(G) + (1 - q_2(G))|G| = |G| + (2 - |G|)q_2(G),$$

where strictness uses $q_0(G) > 0$. Hence

$$q_2(G) < \frac{|G|/2}{|G| - 2} = \frac{1}{2} + \frac{1}{|G| - 2}.$$

So $q_2(G) < 3/4$ for $|G| \ge 6$. If $|G| \le 5$, use split graph G_s: if G has a bridge then $q_2(G) \le 1/2$ by item 3. Otherwise G is biconnected with at most 5 edges, hence $G \in \{K_3, C_4, C_5, K_4^-\}$, and direct checks give

$$q_2(K_3) = q_2(C_4) = \frac{3}{4}, \quad q_2(C_5) = \frac{5}{8}, \quad q_2(K_4^-) = \frac{1}{4}.$$

□

remark. Lemma 3.1 提供了 *Claim 2* 所需的“统一概率上界工具箱”，特别是 $q_2 \le 3/4$ 这条在偶数边情形里反复使用。

Lemma 3.2 (Lemma 3.2 (Original))**.** *Let G be a graph, and let H be the union of its biconnected components.*

1. $$q_0(\emptyset) = 1, \qquad q_0(-) = \frac{1}{2}, \qquad q_0(G) \le \frac{1}{4}$$
   for all other graphs.

2. *If $m = 0$ and $|G|$ is odd, then either $q_0(G) \le 1/16$, or G is a triangle or K_4^-.*

3. *Either $H = \emptyset$, or $a_0 \le 1/4$.*

Proof.

1. Immediate from Lemma 3.1(1).

2. Since $m = 0$, each component is biconnected (thus has at least 3 vertices). If at least two components, Lemma 3.1(1) gives $q_0(G) \le 1/16$. So G is connected. If $v(G) \ge 5$, again $q_0(G) \le 1/16$. Remaining odd-edge possibilities are K_3 and K_4^-.

3. H is a union of biconnected graphs, hence $H \ne -$. Apply item 1.

□

remark. Lemma 3.2 的作用是"压缩异常图名单"，把很多情况归结为少数可直接验算的特殊图。

Proof of Claim 2 (Original). Write

$$f(G) = q_0(G) - \frac{5}{7}q_1(G) - \frac{1}{7}q_2(G) + \frac{3}{28}q_3(G).$$

The proof splits into odd $|G|$ and even $|G|$.

Odd number of edges. We show: if $|G|$ is odd, then

$$f(G) \le \frac{1}{7},$$

with equality iff G is an edge, a triangle, or K_4^-; otherwise

$$f(G) \le \frac{1}{7} - \frac{1}{56}.$$

If G has m bridges, Lemma 3.1 gives $q_1 = mq_0$, so

$$f(G) = \left(1 - \frac{5}{7}m\right) q_0(G) - \frac{1}{7}q_2(G) + \frac{3}{28}q_3(G). \tag{5}$$

If $m = 1$, then

$$f(G) = \frac{2}{7}q_0(G) - \frac{1}{7}q_2(G) + \frac{3}{28}q_3(G).$$

If $G = -$, then $f(G) = -1/7$. Otherwise, by Lemma 3.2(1), $q_0(G) \le 1/4$, and by Lemma 3.1(4), $q_3(G) \le 1/2$, hence

$$f(G) \le \frac{2}{7} \cdot \frac{1}{4} + \frac{3}{28} \cdot \frac{1}{2} = \frac{1}{8} = \frac{1}{7} - \frac{1}{56}.$$

If $m \ge 2$, the coefficient of q_0 in (5) is negative, so

$$f(G) < \frac{3}{28} = \frac{1}{7} - \frac{1}{28} < \frac{1}{7} - \frac{1}{56}.$$

Now assume $m=0$. If $q_0(G) \le 1/16$, then with $q_3(G) \le 1/2$,

$$f(G) \le \frac{1}{16} + \frac{3}{28}\cdot\frac{1}{2} = \frac{13}{112} = \frac{1}{7} - \frac{3}{112} < \frac{1}{7} - \frac{1}{56}.$$

Otherwise, Lemma 3.2(2) implies G is either K_3 or K_4^-, and direct computation gives

$$f(K_3) = f(K_4^-) = \frac{1}{7}.$$

Even number of edges. We show: if $|G|$ is even, then

$$f(G) \ge -\frac{1}{7},$$

with equality iff G is a 2-forest or a 4-forest; and otherwise

$$f(G) \ge -\frac{1}{7} + \frac{1}{28}.$$

By (4),

$$\begin{aligned} f(G) &= \frac{1}{2^m}\Big[a_0 - \frac{5}{7}ma_0 - \frac{1}{7}\left(\binom{m}{2}a_0 + a_2\right) + \frac{3}{28}\left(\binom{m}{3}a_0 + ma_2 + a_3\right)\Big] \\ &= \frac{1}{2^m}\Big[\left(1 - \frac{5}{7}m - \frac{1}{7}\binom{m}{2} + \frac{3}{28}\binom{m}{3}\right)a_0 + \left(-\frac{1}{7} + \frac{3m}{28}\right)a_2 + \frac{3}{28}a_3\Big]. \end{aligned}$$

When $m=0$,

$$f(G) = a_0 - \frac{1}{7}a_2 + \frac{3}{28}a_3.$$

By Lemma 3.1(5), $a_2 \le 3/4$, so

$$f(G) > -\frac{1}{7} + \frac{1}{28}.$$

When $m=1$,

$$f(G) = \frac{1}{2}\left(\frac{2}{7}a_0 - \frac{1}{28}a_2 + \frac{3}{28}a_3\right) = \frac{1}{7}a_0 - \frac{1}{56}a_2 + \frac{3}{56}a_3 > -\frac{3}{4}\cdot\frac{1}{56} = -\frac{1}{7} + \frac{29}{224} > -\frac{1}{7} + \frac{1}{28}.$$

When $m=2$,

$$f(G) = \frac{1}{4}\left(-\frac{4}{7}a_0 + \frac{1}{14}a_2 + \frac{3}{28}a_3\right) = -\frac{1}{7}a_0 + \frac{1}{56}a_2 + \frac{3}{112}a_3.$$

We have $f(G) = -1/7$ **iff** $H = \emptyset$, i.e. G has exactly two edges. If $H \ne \emptyset$, Lemma 3.2(3) gives $a_0 \le 1/4$, hence

$$f(G) \ge -\frac{1}{28} = -\frac{1}{7} + \frac{3}{28} > -\frac{1}{7} + \frac{1}{28}.$$

When $m=3$,

$$f(G) = \frac{1}{8}\left(-\frac{41}{28}a_0 + \frac{5}{28}a_2 + \frac{3}{28}a_3\right) = -\frac{41}{224}a_0 + \frac{5}{224}a_2 + \frac{3}{224}a_3.$$

Since $|G|$ is even, $H \ne \emptyset$, so $a_0 \le 1/4$ and

$$f(G) \ge -\frac{41}{896} = -\frac{1}{7} + \frac{87}{896} > -\frac{1}{7} + \frac{1}{28}.$$

When $m=4$,

$$f(G) = \frac{1}{16}\left(-\frac{16}{7}a_0 + \frac{2}{7}a_2 + \frac{3}{28}a_3\right) = -\frac{1}{7}a_0 + \frac{1}{56}a_2 + \frac{3}{448}a_3.$$

We have $f(G) = -1/7$ **iff** $H = \emptyset$, i.e. G is a 4-forest. Otherwise $a_0 \le 1/4$, so

$$f(G) \ge -\frac{1}{28} = -\frac{1}{7} + \frac{3}{28} > -\frac{1}{7} + \frac{1}{28}.$$

Finally, assume $m \ge 5$. Since coefficients of a_2, a_3 are positive for $m \ge 2$, it suffices to bound the a_0-coefficient away from $-1/7$. Define

$$r(m) = \frac{1}{2^m}\left(1 - \frac{5}{7}m - \frac{1}{7}\binom{m}{2} + \frac{3}{28}\binom{m}{3}\right).$$

For $m = 5$,

$$r(5) = -\frac{41}{448}.$$

Since $|G|$ is even, $H \ne \emptyset$, so $a_0 \le 1/4$ and

$$f(G) \ge -\frac{41}{448} \cdot \frac{1}{4} = -\frac{1}{7} + \frac{215}{1792} > -\frac{1}{7} + \frac{1}{28}.$$

For $m = 6$,

$$r(6) = -\frac{23}{448}, \qquad f(G) \ge -\frac{23}{448} = -\frac{1}{7} + \frac{41}{448} > -\frac{1}{7} + \frac{1}{28}.$$

For $m = 7$,

$$r(7) = -\frac{13}{512}.$$

For $m \ge 7$, the polynomial

$$1 - \frac{5}{7}m - \frac{1}{7}\binom{m}{2} + \frac{3}{28}\binom{m}{3}$$

in the numerator of $r(m)$ is strictly increasing, hence

$$r(m) \ge -\frac{13}{512} \qquad (\forall\, m \ge 7).$$

Therefore

$$f(G) \ge -\frac{13}{512} = -\frac{1}{7} + \frac{421}{3584} > -\frac{1}{7} + \frac{1}{28}$$

for all $m \ge 7$. This completes the proof of Claim 2. $\square$

remark. ***Section 3* 核心收获**: *Claim 2* 所需的最小特征值与谱间隙都由 *cut* 统计精确推出，且等号图形态被完整刻画。

难点（对应此处）：为什么要按桥数 m 分这么细？

remark. 因为式 *(4)* 下 m 直接改变 a_0, a_2, a_3 的系数符号和大小；不分 m 做分段估计，很难同时控制下界与等号情形。

4 Section 4 Notes

Section Transition (Original). For $p < 1/2$, the intersecting and agreeing questions are no longer equivalent. Indeed, the triangle-agreeing family of all graphs containing no edges of a fixed triangle has

$$\mu_p(\mathcal{F}) = (1-p)^3 > p^3.$$

So in this section we focus only on odd-cycle-intersecting families.

remark. 这一步把讨论对象从 "*intersecting/agreeing* 二者可互转" 切换到 "只做 *intersecting*"，是偏置区间最关键的结构变化。

4.1 Skew Analysis

Weighted Cube Setting (Original). For skew analysis, work on $\{0,1\}^X$ (finite X) with product measure

$$\mu_p(S) = p^{|S|}(1-p)^{|X|-|S|}.$$

In our graph setting, $X = [n]^{(2)}$. For $G \subseteq K_n$:

$$\mu_p(G) = p^{|G|}(1-p)^{\binom{n}{2}-|G|}, \qquad \mu_p(\mathcal{F}) = \sum_{G\in\mathcal{F}} \mu_p(G).$$

Inner product:

$$\langle f, g\rangle_p = \sum_{S\subseteq X} \mu_p(S) f(S) g(S).$$

remark. 这一步把均匀测度的 *Hilbert* 结构替换成偏置版本，后续所有正交分解都随之 *skew* 化。

Skew Fourier–Walsh Basis (Original). For each $e \in X$, define

$$\chi_e(S) = \begin{cases} \sqrt{\frac{p}{1-p}}, & e \notin S, \\ -\sqrt{\frac{1-p}{p}}, & e \in S. \end{cases}$$

For $R \subseteq X$, let $\chi_R = \prod_{e\in R} \chi_e$. Then $\{\chi_R : R \subseteq X\}$ is an orthonormal basis in $(\mathbb{R}[\{0,1\}^X], \langle\cdot,\cdot\rangle_p)$. Every f has expansion

$$f = \sum_{R\subseteq X} \widehat{f}(R)\chi_R.$$

All formulas from Section 2.1 (Parseval etc.) hold in this skew setting.

remark. 技术上只换了基和内积，但证明主线不变：依然是“构造 *OCC* 算子 + 频谱约束”。

> **Definition 4.1** (Definition 4.1 (Skew OCC Operator, Original)). *For $p < 1/2$, an OCC operator is a linear operator A on $\mathbb{R}[\{0,1\}^{[n]^{(2)}}]$ such that:*
>
> 1. *if f is the indicator of an odd-cycle-intersecting family, then*
>
> $$f(G) = 1 \Rightarrow Af(G) = 0;$$
>
> 2. *the skew Fourier–Walsh basis is a complete eigenbasis of A.*

remark. 和 $p = 1/2$ 的定义唯一关键区别是这里用的是 ***intersecting***（不是 *agreeing*）。

Matrix Construction (Original). Define

$$M = \begin{pmatrix} \frac{1-2p}{1-p} & \frac{p}{1-p} \\ 1 & 0 \end{pmatrix},$$

with rows/columns indexed by $\{0,1\}$. For bipartite $B \subseteq K_n$, set for each edge e:

$$M_B^{(e)} = \begin{cases} M, & e \in B, \\ I_{2\times 2}, & e \notin B, \end{cases}$$

and

$$M_B = \bigotimes_{e \in K_n} M_B^{(e)}.$$

remark. 这就是 *skew* 版的"翻边算子工程模板"：把一维矩阵张量化到所有边坐标。

Claim 4.1 (Claim 4 (Original)). *For bipartite B, left multiplication by M_B defines an OCC operator:*

$$(M_B f)(G) = \sum_{H \subseteq K_n} (M_B)_{G,H} f(H).$$

For each $G \subseteq K_n$, χ_G is an eigenvector with eigenvalue

$$\lambda_G = \left(-\frac{p}{1-p}\right)^{|G \cap B|} = \left(-\frac{p}{1-p}\right)^{|G|} \left(-\frac{1-p}{p}\right)^{|G \cap B|}.$$

Proof. **annihilation step:** if $\mathcal{F}$ is odd-cycle-intersecting with indicator f, to show OCC property it suffices by linearity to prove

$$G \cap H \cap B \neq \emptyset \Rightarrow (M_B)_{G,H} = 0.$$

But

$$(M_B)_{G,H} = \prod_{e \in K_n} (M_B^{(e)})_{G(e),H(e)},$$

and if some $e \in B$ has $G(e) = H(e) = 1$, the corresponding factor is $M_{1,1} = 0$.

eigenvalue step: the vectors

$$\chi_\emptyset = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad \chi_{\{e\}} = \begin{pmatrix} \sqrt{p/(1-p)} \\ -\sqrt{(1-p)/p} \end{pmatrix} \tag{6}$$

are simultaneous eigenvectors of M and $I_{2\times 2}$. Tensorization gives that each χ_G is an eigenvector of M_B with eigenvalue

$$\left(-\frac{p}{1-p}\right)^{|G \cap B|},$$

equivalently in the displayed claim form. □

remark. *Claim 4* 把 *Section 2* 的算子路线完整迁移到了 *skew* 场景，是 *Section 4* 的基础支点。

Uniform-Case Consistency (Original). M_B is the p-skew analogue of the uniform operator A_B; when $p = 1/2$, we have $M_{0,0} = 0$, hence

$$M_B = A_B.$$

remark. 这一步说明 *Section 4* 不是另起炉灶，而是对 *Section 2* 的连续变形。

Transpose/Random-Walk View (Original). $M_B^\top$ corresponds to an operator N_B:

$$N_B f(G) = \mathbb{E}[f(G \oplus_p B)],$$

where $G \oplus_p B$ means flipping each edge of B independently with probability $p/(1-p)$. This is a Markov walk with stationary law $G(n,p)$, and for odd-cycle-intersecting f:

$$f(G) = 1 \Rightarrow N_B f(G) = 0.$$

remark. 这个概率过程视角解释了为什么偏置模型下仍能保留“相邻一步不可能都落在族内”的核心机制。

4.2 Engineering Eigenvalues For $p < 1/2$

Subsection Aim (Original). In this subsection, we construct an OCC operator with the required spectrum first on $[1/4, 1/2)$, and in fact on a slightly extended interval, so that the stability constant remains bounded when p is bounded away from 0.

remark. 这说明作者目标不只是“能证上界”，还要把稳定性常数一起控制住。

Working Interval (Original). In this subsection, assume

$$p \in [\tau, 1/2), \qquad \tau = 0.248.$$

The proof for smaller p is deferred to Section 4.4. *The proof breaks down for slightly smaller p: the required inequality is violated by 3-forests.*

remark. 作者先在“中等偏置区间”把谱构造做稳，再用另一套谱覆盖小 p。

Lemma 4.2 (Lemma 4.2 (Original)). *Let $\mathcal{B}$ be a distribution over bipartite graphs, and each $B \in \mathcal{B}$ has a function f_B on subgraphs of B. Then*

$$\lambda_G = \left(-\frac{p}{1-p}\right)^{|G|} \mathbb{E}[f_B(B \cap G)]$$

is an OCC spectrum.

Proof. Trivial skew generalization of the proof of Lemma 2.8. □

remark. 这一条是 *skew* 版本的“谱工厂”。

Corollary 4.3 (Corollary 4.3 (Original)). *With random bipartition (V_1, V_2) and cut B, define*

$$q_i(G) = \Pr(|G \cap B| = i), \qquad q_R(G) = \Pr(G \cap B \cong R).$$

Then

$$\lambda_G = \left(-\frac{p}{1-p}\right)^{|G|} q_i(G), \qquad \lambda_G = \left(-\frac{p}{1-p}\right)^{|G|} q_R(G)$$

are OCC spectra.

remark. Corollary 4.3 表示 *Section 2* 的 q_i, q_R 工具箱在 *skew* 情形可直接复用。

Theorem 4.4 (Theorem 4.4 (Original))**.** *Let* $\Lambda = (\lambda_G)_{G \subseteq K_n}$ *be an OCC spectrum with*

$$\lambda_\emptyset = 1, \quad -1 < \lambda_{\min} < 0, \quad \gamma > 0, \quad \nu = -\frac{\lambda_{\min}}{1 - \lambda_{\min}}.$$

For odd-cycle-intersecting $\mathcal{F}$*:*

- *Upper bound:* $\mu_p(\mathcal{F}) \leq \nu$.
- *Uniqueness: if* $\mu_p(\mathcal{F}) = \nu$*, then* $\widehat{\mathcal{F}}(G) \neq 0$ *only on* $\Lambda_{\min} \cup \{\emptyset\}$.
- *Stability:*
$$w := \sum_{G \notin \Lambda_{\min} \cup \{\emptyset\}} \widehat{\mathcal{F}}(G)^2 \leq \frac{\nu}{(1-\nu)\gamma} (\nu - \mu_p(\mathcal{F})).$$

Proof. Same argument as Theorem 2.2, replacing uniform Fourier analysis by the p-skew Fourier basis and using odd-cycle-intersecting OCC annihilation. □

remark. Theorem 4.4 是 *Section 2.2* 主模板在 *skew* 空间里的一一对应版。

Corollary 4.5 (Corollary 4.5 (Original))**.** *Assume an OCC spectrum with*

$$\lambda_\emptyset = 1, \qquad \lambda_{\min} = -\frac{p^3}{1-p^3}, \qquad \gamma > 0,$$

and all minimizers have at most 3 edges. If $\mathcal{F}$ *is odd-cycle-intersecting, then:*

- $\mu_p(\mathcal{F}) \leq p^3$;
- *if equality,* $\mathcal{F}$ *is a* $\triangle$*umvirate;*
- *if* $\mu_p(\mathcal{F}) > p^3 - \varepsilon$*, then*
$$\mu_p(\mathcal{F} \triangle \mathcal{T}) = O_p\left(\frac{p^3}{(1-p^3)\gamma}\varepsilon\right)$$
for some $\triangle$*umvirate* $\mathcal{T}$.

Proof. Same deduction as Corollary 2.3 from Theorem 2.2, now using Theorem 4.4. □

remark. Corollary 4.5 是 *Section 4* 的直接目标条件：只要构造出正确最小特征值和间隙，主结论就落地。

Coefficient Constraints (Original). For spectra of the form

$$\lambda_G = \left(-\frac{p}{1-p}\right)^{|G|} \left[c_0 q_0(G) + c_1 q_1(G) + c_2 q_2(G) + c_3 q_3(G) + c_4 q_4(G)\right],$$

the small-graph constraints give

$$c_0 = 1, \qquad c_1 = \frac{p^2 - p - 1}{p^2 + p + 1}, \qquad c_2 = \frac{p^2 - 3p + 1}{p^2 + p + 1},$$

and

$$\frac{5p^2 - 27p + 45 - 16/p}{p^2 + p + 1} \le 4c_3 + c_4 \le \frac{5p^2 - 27p + 45 - 32/p + 8/p^2}{p^2 + p + 1}.$$

At $p = 1/2$ bounds coincide; for $p > 1/2$ they contradict; for $p < 1/2$ they leave a gap. Choose

$$c_4 = 0, \qquad c_3 = \frac{5p^2 - 27p + 45 - 28/p + 6/p^2}{4(p^2 + p + 1)},$$

which satisfies $c_3 > 0$ on $(0, 1/2]$.

remark. 这一步正是 *Section 3* 里 "*good fortune at* $p = 1/2$" 在 *skew* 下的精确对应：$p > 1/2$ 会直接失效。

Claim 4.6 (Claim 5 (Original)). *Let* $\Lambda^{(1)}$ *be given by*

$$\lambda_G^{(1)} = \left(-\frac{p}{1-p}\right)^{|G|} \left[q_0(G) + c_1 q_1(G) + c_2 q_2(G) + c_3 q_3(G)\right],$$

with c_1, c_2, c_3 *as above. Then there exists* $\gamma' > 0$ *(independent of* n, p*) such that:*

- $\lambda_\emptyset^{(1)} = 1$;
- $\lambda_{\min}^{(1)} = -\frac{p^3}{1-p^3}$;
- *minimizers are: one edge, 2-path, two disjoint edges, triangle;*
- *for all*

  $$H \notin \Lambda_{\min}^{(1)} \cup \mathcal{F}_4 \cup \{K_4^-\},$$

  we have

  $$\lambda_H^{(1)} \ge -\frac{p^3}{1-p^3} + \gamma'.$$

remark. *Claim 5* 是 *skew* 第一主谱：目标点命中，非目标点留间隙，但 *4-forest* 与 K_4^- 仍需二次修正。

Claim 4.7 (Claim 6 (Original)). *Define*

$$\lambda_G^{(2)} = \left(-\frac{p}{1-p}\right)^{|G|} \left[\sum_{F \in \mathcal{F}_4} q_F(G) - q_{C_4}(G)\right].$$

Then

1. $\lambda_H^{(2)} = 0$ *for* $|H| < 4$;
2. $\lambda_F^{(2)} = 2^{-4} \frac{p^4}{(1-p)^4}$ *for 4-forests* F;
3. $\lambda_{K_4^-}^{(2)} = 2^{-3} \frac{p^5}{(1-p)^5}$;
4. $|\lambda_G^{(2)}| \le 1$ *for all* G.

Proof. Same proof as Claim 3; the last bound uses $\left|\frac{p}{1-p}\right| \le 1$. $\square$

remark. *Claim 6 是 Claim 3 的 skew 复制件，专门把 4-edge 异常 tight 点抬高。*

Corollary 4.8 (Corollary 4.6 (Original)). *Let*

$$\Lambda = \Lambda_1 + \frac{16}{17}\gamma'\Lambda_2.$$

Then:

- $\lambda_\emptyset = 1$;
- $\lambda_G = -\frac{p^3}{1-p^3}$ *on non-empty subgraphs of* K_3 *(and two disjoint edges);*
- *with*

$$\gamma = \frac{p^4}{17(1-p)^4}\gamma' \geq \frac{\tau^4}{17(1-\tau)^4}\gamma',$$

  all $|G| > 3$ *satisfy*

$$\lambda_G \geq -\frac{p^3}{1-p^3} + \gamma.$$

Proof. Same linear-combination argument as Corollary 2.10, with skew scaling factors. □

remark. *Corollary 4.6 完成了* $p \in [\tau, 1/2)$ *的谱条件闭环。*

Immediate Consequence (Original). Corollary 4.6 implies Theorem 1.4 for $p \in [\tau, 1/2)$. The remaining range is handled in Subsection 4.4.

remark. *到这里中等偏置区间已经收工，后面只需补上小* p *区间。*

4.3 Proof Of Claim 5

Verification Strategy (Original). The proof follows Claim 2 structure (odd/even $|G|$ cases), but now inequalities are rational in p. Each target inequality is reduced to polynomial positivity on $[\tau, 1/2)$, checked via Sturm-chain style root exclusion.

remark. *这段说明了"为何可一次性覆盖整个区间"：不是点代入，而是函数级别验证。*

Lemma 4.9 (Lemma 4.7 (Original)). *Let* G *be a graph with* m *bridges.*

1. *If* $m = 1$ *and* $|G| > 1$, *then* $|G| \geq 4$.
2. *If* $m = 0$ *and* $|G| \leq 5$, *then* G *is one of* K_3, C_4, C_5, K_4^-.

Proof.

1. Every biconnected component has at least 3 edges.
2. If there are two biconnected components then $|G| \geq 6$; the only biconnected graphs with at most 5 edges are those listed.

□

remark. *Lemma 4.7 是 Claim 5 分情况估计时的有限分类入口。*

Proof of Claim 5 (Original). Set coefficients as above. For $p \in (0, 1/2]$, $c_0, c_3 > 0$, $c_1 < 0$, and c_2 changes sign at $(3 - \sqrt{5})/2$.

Odd $|G|$ case. By Lemma 3.1(4,5):

$$\lambda_G \geq -\left(\frac{p}{1-p}\right)^{|G|}\left[q_0(1 + mc_1) + \max\left(\frac{3}{4}c_2, 0\right) + \frac{1}{2}c_3\right].$$

When $m \geq 2$, $1 + mc_1 < 0$, so

$$\lambda_G \geq -\left(\frac{p}{1-p}\right)^{|G|}\left[\max\left(\frac{3}{4}c_2, 0\right) + \frac{1}{2}c_3\right].$$

If $|G| = 3$, direct check gives $\lambda_G > -\frac{p^3}{1-p^3}$ unless G is intended minimizer. Otherwise $|G| \geq 5$ and

$$\lambda_G \geq -\left(\frac{p}{1-p}\right)^5\left[\max\left(\frac{3}{4}c_2, 0\right) + \frac{1}{2}c_3\right] > -\frac{p^3}{1-p^3}.$$

When $m = 1$, Lemma 4.7(1): either $|G| = 1$ (equality minimizer) or $|G| \geq 5$. Using $q_0 \leq 1/4$:

$$\lambda_G \geq -\left(\frac{p}{1-p}\right)^5\left[\frac{1}{4}(1 + c_1) + \max\left(\frac{3}{4}c_2, 0\right) + \frac{1}{2}c_3\right] > -\frac{p^3}{1-p^3}.$$

When $m = 0$, Lemma 4.7(2): either K_3, C_5, K_4^- or $|G| \geq 7$. K_3 gives equality minimizer; C_5, K_4^- are checked directly (for K_4^- equality only at $p = 1/2$). For $|G| \geq 7$, Lemma 3.2(2) gives $q_0 \leq 1/16$:

$$\lambda_G \geq -\left(\frac{p}{1-p}\right)^7\left[\frac{1}{16} + \max\left(\frac{3}{4}c_2, 0\right) + \frac{1}{2}c_3\right] > -\frac{p^3}{1-p^3}.$$

Even $|G|$ case. Using (4), write

$$\lambda_G = \left(\frac{p}{1-p}\right)^{|G|}\big(d_0(m)a_0 + d_2(m)a_2 + d_3(m)a_3\big),$$

where

$$d_0(m) = 2^{-m}\left[1 + mc_1 + \binom{m}{2}c_2 + \binom{m}{3}c_3\right], \quad d_2(m) = 2^{-m}(c_2 + mc_3), \quad d_3(m) = 2^{-m}c_3.$$

Here $d_3(m) > 0$, and $d_2(m) > 0$ for $m \geq 2$. Also $d_0(m) > 0$ for $m \geq 10$ (via positivity of $c_1 + 7c_3$, $c_2 + 2c_3$, and monotonic increment argument). Hence if $m \geq 10$, $\lambda_G > 0$.

If $m < 10$ and G is a forest, possible even-edge cases are 2-,4-,6-,8-forests. 2-forest gives equality minimizer; others are direct checks (4-forest touches only at $p = 1/2$).

If $m < 10$ and G is not a forest, Lemmas 3.1(5), 3.2(3) imply

$$\lambda_G \geq \left(\frac{p}{1-p}\right)^2\left[\min\left(\frac{1}{4}d_0(m), 0\right) + \min\left(\frac{3}{4}d_2(m), 0\right)\right] > -\frac{p^3}{1-p^3}.$$

So all non-target graphs are strictly above the minimum, yielding the claimed gap. ☐

remark. Claim 5 的关键是：*odd/even* 两条链都能统一压到目标阈值之上，并且在区间上保持严格性。

4.4 Small p

Range (Original). For $p \in (0, \tau]$, use a different OCC spectrum read from [8].

remark. 这是 *Section 4* 的第二套方案，用来填补主构造在更小 p 区间的技术空档。

Claim 4.10 (Claim 7 (Original)). *An OCC spectrum is*

$$\lambda_G = \lambda_{|G|} = \left(-\frac{p}{1-p}\right)^{|G|}\left[1 - \frac{1+p}{1+p+p^2}|G| + \frac{1}{1+p+p^2}\binom{|G|}{2}\right].$$

Moreover:

1. $\lambda_0 = 1$;
2. $\lambda_{\min} = \lambda_1 = \lambda_2 = \lambda_3 = -\frac{p^3}{1-p^3}$;
3. *for* $|G| \geq 4$,
$$\lambda_{|G|} \geq \lambda_5 = -\left(\frac{p}{1-p}\right)^5 \frac{6-4p+p^2}{1+p+p^2},$$
so the spectral gap is
$$\gamma = \frac{p^3}{1-p^3} - \left(\frac{p}{1-p}\right)^5 \frac{6-4p+p^2}{1+p+p^2}.$$

Proof. Can be deduced from [8]. Alternatively, by Lemma 4.2, any

$$\lambda_G = \left(-\frac{p}{1-p}\right)^{|G|}\left(a_0 + a_1|G| + a_2\binom{|G|}{2}\right)$$

is OCC. Choose coefficients to satisfy $\lambda_0 = 1$ and

$$\lambda_1 = \lambda_2 = \lambda_3 = -\frac{p^3}{1-p^3}.$$

Then direct checks show:

$$\lambda_5 < 0, \qquad \lambda_{|G|} \geq 0 \ (|G| \geq 4 \text{ even}), \qquad |\lambda_{|G|+2}| < |\lambda_{|G|}| \ (|G| \geq 5 \text{ odd}),$$

which yields the claim. □

remark. Claim 7 在小 p 区间给出可显式控制的单变量谱。

Lemma 4.11 (Lemma 4.8 (Original)). *Let γ be the spectral gap in Claim 7. Then*

$$\frac{p^3}{(1-p^3)\gamma}$$

is bounded on $p \in (0, \tau]$.

Proof. Let
$$g(p) = \frac{(1-p^3)\gamma}{p^3} = 1 - \frac{p^2(6-4p+p^2)}{(1-p)^4}.$$
On $[0, 1/4]$, g is strictly decreasing with $g(0) = 1$, $g(1/4) = 0$. Hence on $[0, \tau]$:

$$g(p) \geq g(\tau) > 0,$$

so

$$\frac{p^3}{(1-p^3)\gamma} \leq \frac{1}{g(\tau)}.$$

□

remark. 这个有界性是稳定性常数不会在小 p 爆炸的关键保障。

Theorem 4.12 (Theorem 4.9 (Kindler–Safra, Original)). *For every $t \in \mathbb{N}$ and $p \in (0,1)$, there exist*

$$\varepsilon_0 = \Omega(p^{4t}), \quad c = O(p^{-t}), \quad T = O(tp^{-4t})$$

such that if Boolean $f : \{0,1\}^N \to \{0,1\}$ satisfies

$$\sum_{|S|>t} \widehat{f}(S)^2 = \varepsilon < \varepsilon_0,$$

then there exists Boolean g depending on at most T coordinates with

$$\mu_p(\{R : f(R) \neq g(R)\}) \leq c\varepsilon.$$

remark. 这条是 *skew* 稳定性闭环的最后分析输入。

From Spectral Tail To Structural Closeness (Original). Lemma 4.8 and Corollary 4.6 imply that there exists an absolute constant C such that if $\mathcal{F}$ is odd-cycle-intersecting with $\mu_p(\mathcal{F}) \geq p^3 - \varepsilon$, then

$$\sum_{|G|>3} \widehat{\mathcal{F}}(G)^2 \leq C\varepsilon.$$

Applying Theorem 4.9 (with $t = 3$) and the same deduction as in Corollary 2.3 yields that $\mathcal{F}$ is **$c_p C\varepsilon$-close** to a $\triangle$umvirate, where c_p depends only on p and is bounded on $[\delta, 1/2)$ for each fixed $\delta > 0$.

remark. 这段把"谱尾小"真正转成"结构接近三角形族"，是稳定性结论落地的最后一步。

Section 4 Conclusion (Original). Combining Corollary 4.6 (for $p \in [\tau, 1/2)$), Claim 7–Lemma 4.8 (for $p \in (0, \tau]$), and Theorem 4.9 gives Theorem 1.4 for all $p \in (0, 1/2)$, with stability constants bounded away from 0 on $[\delta, 1/2)$.

remark. ***Section 4* 核心收获**：偏置测度下的极值值 p^3、结构唯一性与稳定性全部打通。

5 Section 5 Notes

5.1 Odd-Linear-Dependency Families

Section Goal (Original). In this section we prove Theorem 1.8.

remark. 这一节把"奇环交"推广到"奇线性依赖交"，并保持同一套谱方法。

Odd-LD-Intersecting On Hypergraphs (Original). A family $\mathcal{F}$ of hypergraphs on $[n]$ is odd-linear-dependency-intersecting (odd-LD-intersecting) if for any $G, H \in \mathcal{F}$, there exist $l \in \mathbb{N}$ and non-empty sets

$$A_1, \ldots, A_{2l+1} \in G \cap H$$

such that

$$A_1 \triangle \cdots \triangle A_{2l+1} = \emptyset.$$

remark. 这相当于要求交集里存在“奇数条关系相加为零”的依赖结构。

Vector-Space Reformulation (Original). Identifying subsets of $[n]$ with vectors in $\{0,1\}^n = \mathbb{Z}_2^n$, symmetric difference becomes vector addition. Equivalently, a family $\mathcal{F}$ of subsets of $\mathbb{Z}_2^n$ is odd-LD-intersecting if for any $S, T \in \mathcal{F}$ there exist $l \in \mathbb{N}$ and non-zero vectors

$$v_1, \ldots, v_{2l+1} \in S \cap T$$

such that

$$v_1 + \cdots + v_{2l+1} = 0.$$

remark. 这里把组合定义变成了线性代数条件，后续傅里叶与 *Cayley* 分析自然可用。

Odd-LD-Agreeing And The Zero Vector (Original). Similarly, odd-LD-agreeing is defined by replacing $S \cap T$ with $S \triangle T$. Since 0 cannot appear in a non-trivial odd linear dependency, it is irrelevant; thus we work on

$$\{0,1\}^n \setminus \{0\} = \mathbb{Z}_2^n \setminus \{0\}.$$

remark. 剔除 0 向量后，结构更整齐，且不影响极值问题本质。

Skew Measure And Schur Structures (Original). For $p \in [0,1]$ and $S \subset \mathbb{Z}_2^n \setminus \{0\}$,

$$\mu_p(S) = p^{|S|}(1-p)^{2^n-1-|S|}, \qquad \mu_p(\mathcal{F}) = \sum_{S \in \mathcal{F}} \mu_p(S).$$

We mainly use $\mu_{1/2}$, denoted by μ. A Schur triple is $\{x, y, z\}$ with $x+y=z$ (equivalently $x+y+z=0$). A Schur junta is defined by prescribing intersection with a fixed Schur triple, and a Schur-umvirate consists of all sets containing a fixed Schur triple.

remark. Schur triple 在这里扮演三角形的角色；*junta/umvirate* 则是对应的极值结构候选。

Theorem 5.1 (Theorem 5.1 (Original)). *If $\mathcal{F}$ is an odd-LD-agreeing family of subsets of $\mathbb{Z}_2^n \setminus \{0\}$, then*

$$\mu(\mathcal{F}) \le \frac{1}{8}.$$

Equality holds if and only if $\mathcal{F}$ is a Schur junta. Moreover, there exists a constant c such that for any $\varepsilon > 0$, if

$$\mu(\mathcal{F}) > \frac{1}{8} - \varepsilon,$$

then there exists a Schur junta $\mathcal{T}$ such that

$$\mu(\mathcal{T} \triangle \mathcal{F}) \le c\varepsilon.$$

remark. 这是均匀测度下的完整“上界 + 唯一性 + 稳定性”结论。

Theorem 5.2 (Theorem 5.2 (Original)) ***[Extremal families]*** *Let $p \le 1/2$. If $\mathcal{F}$ is odd-LD-intersecting in $\mathbb{Z}_2^n \setminus \{0\}$, then*

$$\mu_p(\mathcal{F}) \le p^3.$$

Equality holds if and only if $\mathcal{F}$ is a Schur-umvirate.

[Stability] *There exists a constant c such that for any $\varepsilon \ge 0$, if*

$$\mu_p(\mathcal{F}) \ge p^3 - \varepsilon,$$

then there exists a Schur-umvirate $\mathcal{T}$ such that

$$\mu_p(\mathcal{T} \triangle \mathcal{F}) \le c\varepsilon.$$

remark. 这是偏置测度下的对应结论，数值上界变成 p^3。

Lifting Back To Graph Families (Original). Theorem 1.4 follows from Theorem 5.2 by lifting graph families: for odd-cycle-intersecting $\mathcal{F}$ of graphs, define

$$\mathcal{H} = \{F \cup S : F \in \mathcal{F},\ S \subseteq \{0,1\}^n \setminus ([n]^{(2)} \cup \{0\})\}.$$

Then

$$\mu_p(\mathcal{H}) = \mu_p(\mathcal{F}),$$

and $\mathcal{H}$ is odd-LD-intersecting, since odd cycles lift to odd linear dependencies.

remark. 这一步明确了 *Section 5* 和图论主定理之间的双向联系。

Why Section 5 Is Not Redundant (Original). The paper notes that Theorem 5.1 does not seem deducible from Theorem 1.4; it is a bona-fide generalization. The setting is in some ways cleaner, since the ground set and its power set live inside a vector space over $\mathbb{Z}_2$.

remark. 作者强调这里不是“换符号重述”，而是更自然、更统一的母框架。

5.2 Cayley Operators

Preliminaries On Vector Sets (Original). For $S \subset \mathbb{Z}_2^n \setminus \{0\}$, write

$$\operatorname{rank}(S) = \dim(\operatorname{Span}(S)).$$

Define

$$I(S) = \{v \in S : v \notin \operatorname{Span}(S \setminus \{v\})\}, \qquad m(S) = |I(S)|,$$

and

$$J(S) = S \setminus I(S).$$

remark. $I(S)$ 抓“独立骨架”，$J(S)$ 抓“依赖部分”，后面分解公式正是围绕这个拆分。

Bilinear Form And Orthogonal Space (Original). For $x, y \in \mathbb{Z}_2^n$,

$$\langle x, y\rangle = \sum_{i=1}^{n} x_i y_i.$$

For $S \subset \mathbb{Z}_2^n$,

$$S^\perp = \{x \in \mathbb{Z}_2^n : \langle x, v\rangle = 0 \ \forall v \in S\},$$

with

$$\dim(S^\perp) + \dim(\mathrm{Span}(S)) = n.$$

remark. 这给出了解空间维数计算的主工具。

Affine-Subspace Reminder (Original). An affine subspace of V is a set $U + a$ where U is a subspace and $a \in V$. If U has dimension d, then $U + a$ has dimension d. An affine subspace of dimension $n-1$ in an n-dimensional space is an affine hyperplane.

remark. 后面的超平面构造都建立在这个术语上。

Lemma 5.3 (Lemma 5.3 (Original)). *If $S = \{v_1, \dots, v_d\} \subset \mathbb{Z}_2^n$ is linearly independent, then for any $r_1, \dots, r_d \in \{0, 1\}$, the set*

$$A = \{x \in \mathbb{Z}_2^n : \langle v_i, x\rangle = r_i \ \forall i \in [d]\}$$

is a translate of $S^\perp$, and therefore an affine subspace of dimension $n-d$.

Proof. For each $i \in [d]$, choose

$$y_i \in (S \setminus \{v_i\})^\perp \setminus S^\perp.$$

Since $\dim((S \setminus \{v_i\})^\perp) = n - d + 1$ and $\dim(S^\perp) = n - d$, such y_i exists. Moreover,

$$\langle y_i, v_j\rangle = \delta_{i,j}.$$

Let

$$a = \sum_{i=1}^{d} r_i y_i.$$

Then

$$A = a + S^\perp,$$

as required. □

remark. 这是“线性方程组解集 = 一个特解 + 齐次解空间”的 $\mathbb{Z}_2$ 版本。

Lemma 5.4 (Lemma 5.4 (Original)). *An affine hyperplane of the form*

$$A_w = \{v \in \mathbb{Z}_2^n : \langle v, w\rangle = 1\}, \qquad w \in \mathbb{Z}_2^n \setminus \{0\},$$

contains no odd linear dependency.

Proof. If $v_1, \dots, v_{2l+1} \in A_w$, then

$$\left\langle \sum_{i=1}^{2l+1} v_i,\ w \right\rangle = \sum_{i=1}^{2l+1} \langle v_i, w \rangle = 1.$$

Hence

$$\sum_{i=1}^{2l+1} v_i \neq 0.$$

□

remark. 这正是“二分图不含奇环”在线性空间里的同构结论。

Definition 5.1 (Definition 5.5 (Hyperplane Subset, Original)). *A hyperplane subset is a subset of some A_w, where $w \in \mathbb{Z}_2^n$.*

remark. 这是第 5 节里“bipartite graph”的替代对象。

Hyperplane Subsets As Bipartite Analogues (Original). If A is a hyperplane subset, $\mathcal{F}$ is odd-LD-agreeing, and $S \in \mathcal{F}$, then

$$S \oplus A \notin \mathcal{F}.$$

Also, a bipartite graph with bipartition (X, Y) is a subset of A_w by taking $w_i = \mathbf{1}_{\{i \in X\}}$.

remark. 这句把“图上的二分结构”和“向量上的超平面结构”桥接起来了。

Random Hyperplane Slice (Original). For uniform random $w \in \mathbb{Z}_2^n$ and $S \subset \{0,1\}^n$, define

$$Y_S = |S \cap A_w|.$$

This is the analogue of random-cut size in graphs, and if $S \subset [n]^{(2)}$, it is exactly random-cut size of S. Write $Q_S(X)$ for the probability-generating function of Y_S.

remark. 第 3 节的 cut 统计在这里变成了 hyperplane-slice 统计。

OLDC Cayley Setup (Original). We now have a Cayley graph on $\mathbb{Z}_2^{\{0,1\}^n \setminus \{0\}}$ with generating set

$$\{A : \ A \text{ is a hyperplane subset}\}.$$

remark. 这一步对应 Section 2 里从图族问题转成 Cayley 谱问题。

Definition 5.2 (Definition 5.6 (Odd-Linear-Dependency-Cayley Operator, Original)). *A linear operator A on real-valued functions on $\mathbb{Z}_2^{\{0,1\}^n \setminus \{0\}}$ is called OLDC if:*

1. *if $\mathcal{F}$ is odd-LD-agreeing and f is its indicator, then*

$$f(S) = 1 \Rightarrow Af(S) = 0;$$

2. *the Fourier–Walsh basis is a complete eigenbasis of A.*

The eigenvalue vector of an OLDC operator is called an OLDC spectrum.

remark. 这是 *OCC* 概念在 *LD* 场景中的一一对应定义。

Corollary 5.5 (Corollary 5.7 (Original)). *Let w be a uniform random vector in $\mathbb{Z}_2^n$ and*

$$A_w = \{v \in \mathbb{Z}_2^n : \langle v, w\rangle = 1\}.$$

For $S \subset \mathbb{Z}_2^n$, define

$$q_i(S) = \Pr(|S \cap A_w| = i).$$

For any $R \subset \mathbb{Z}_2^n$ with no non-trivial odd linear dependency, define

$$q_R(S) = \Pr(S \cap A_w \cong R),$$

where $\cong$ means linear isomorphism in $\mathbb{Z}_2^n$. Then, for fixed i,

$$\lambda_S = (-1)^{|S|} q_i(S)$$

is an OLDC spectrum, and for fixed admissible R,

$$\lambda_S = (-1)^{|S|} q_R(S)$$

is also an OLDC spectrum.

remark. 这就是 *Section 2* 的 *Corollary 2.9* 在 *LD* 框架中的完全镜像版。

5.3 Construction Of The OLDC Spectrum

Independence Input (Original). If $S = \{v_1, \ldots, v_d\}$ is linearly independent, then $\{\langle v_i, w\rangle : i \in [d]\}$ are independent $\mathrm{Bin}(1, 1/2)$ random variables. Indeed, for any $r_1, \ldots, r_d \in \{0, 1\}$, Lemma 5.3 implies

$$\{w \in \mathbb{Z}_2^n : \langle v_i, w\rangle = r_i \ \forall i \in [d]\}$$

is an affine subspace of dimension $n - d$, hence has size 2^{n-d}.

remark. 这一步提供了"可乘性"，是后面生成函数分解的核心。

Decomposition By $I(S)$ And $J(S)$ (Original). For each $v \in I(S)$, the random variable $\langle v, w\rangle$ is independent of $\{\langle v', w\rangle : v' \in S \setminus \{v\}\}$. So if

$$I(S) = \{v_1, \ldots, v_m\}, \qquad J(S) = \{u_1, \ldots, u_l\},$$

then

$$Y_S = \sum_{i=1}^{m} Y_{\{v_i\}} + Y_{J(S)}.$$

remark. 这个分解是第 *5* 节对应式 *(4)* 的起点。

Generating-Function Expansion (Original). Write

$$Q_{J(S)}(X) = \sum_{i \geq 0} a_i X^i,$$

with $a_1 = 0$. Then

$$\begin{aligned} Q_S(X) &= \left(\frac{1}{2} + \frac{1}{2}X\right)^m Q_{J(S)}(X) \\ &= \frac{1}{2^m}(1+X)^m(a_0 + a_2X^2 + a_3X^3 + \cdots) \\ &= \frac{1}{2^m}\Big(a_0 + ma_0X + (\binom{m}{2}a_0 + a_2)X^2 \\ &\qquad + (\binom{m}{3}a_0 + ma_2 + a_3)X^3 + R(X)X^4\Big), \end{aligned} \tag{7}$$

for some $R(X) \in \mathbb{Q}[X]$.

remark. 式 *(7)* 与 *Section 3* 的式 *(4)* 完全同型，因此 *Claim 2* 的证明框架可以直接平移。

Claim 5.6 (Claim 8 (Original))**.** *Let $\Lambda^{(1)}$ be the OLDC spectrum*

$$\lambda_S^{(1)} = (-1)^{|S|}\left[q_0(S) - \frac{5}{7}q_1(S) - \frac{1}{7}q_2(S) + \frac{3}{28}q_3(S)\right].$$

Then:

- $\lambda_\emptyset^{(1)} = 1$;
- $\lambda_{\text{min}}^{(1)} = -1/7$;
- $\Lambda_{\text{min}}^{(1)}$ *consists of: all singletons, all 2-sets, all linearly independent 4-sets, and all sets of the form* $\{x, y, z, x+y, x+z\}$;
- *for all* $S \notin \Lambda_{\text{min}}^{(1)}$,
$$\lambda_S^{(1)} \geq -\frac{1}{7} + \gamma', \qquad \gamma' = \frac{1}{56}.$$

remark. 其中 *2-sets* 对应 *2-forest*，独立 *4-sets* 对应 *4-forest*，$\{x, y, z, x+y, x+z\}$ 对应 K_4^-。

Lemma 5.7 (Lemma 5.8 (Original))**.** *Let S be a set of vectors.*

1. $$q_0(S) = 2^{-\text{rank}(S)}.$$
2. $$q_1(S) = m(S)q_0(S) = m(S)2^{-\text{rank}(S)}.$$
3. *If there exists $i \in [n]$ such that $|\{v \in S : v(i) = 1\}|$ is odd, then $q_k(S) \leq 1/2$ for any* $k \geq 0$.
4. *For odd k, $q_k(S) \leq 1/2$.*
5. *Always $q_2(S) \leq 3/4$.*

Proof. 1. $q_0(S)$ is the probability that $w \in S^\perp$. Since $\dim(S^\perp) = n - \operatorname{rank}(S)$, we get $q_0(S) = 2^{-\operatorname{rank}(S)}$.

2. $|S \cap A_w| = 1$ iff $w \in (S \setminus \{v\})^\perp \setminus S^\perp$ for some $v \in I(S)$. The sets $\{(S \setminus \{v\})^\perp \setminus S^\perp : v \in I(S)\}$ are disjoint, and each has size $2^{n-\operatorname{rank}(S)}$, giving the formula.

3. Let
$$T_k = \{w \in \{0,1\}^n : |S \cap A_w| = k\}.$$
For any $w \in T_k$, we have $w + e_i \notin T_k$ (same involution idea as Lemma 3.1), hence $|T_k| \leq 2^{n-1}$ and $q_k(S) \leq 1/2$.

4. By item 3, assume each coordinate appears an even number of times in S. Then for any $w \in \mathbb{Z}_2^n$,
$$\sum_{s \in S} \langle s, w \rangle = 0,$$
so $|\{s \in S : \langle s, w \rangle = 1\}|$ is even. Therefore $q_k(S) = 0$ for odd k.

5. Average size gives
$$\frac{|S|}{2} = \sum_k k q_k(S) < 2q_2(S) + (1 - q_2(S))|S| = |S| + (2 - |S|)q_2(S),$$
strictly since $q_0(S) > 0$. Hence
$$q_2(S) < \frac{|S|}{2(|S|-2)} = \frac{1}{2} + \frac{1}{|S|-2}.$$
So $q_2(S) < 3/4$ for $|S| \geq 6$. For $|S| \leq 5$, by item 3 we may assume $\sum_{v \in S} v = 0$. Let $T \subset S$ be the smallest dependent subset. Then T must be
$$\{x, y, x+y\}, \ \{x, y, z, x+y+z\}, \ \text{or } \{x, y, z, v, x+y+z+v\}.$$
Since S sums to 0, also $S \setminus T$ sums to 0; but $|S \setminus T| \leq 2$, impossible unless $S = T$. Directly:
$$q_2(\{x, y, x+y\}) = q_2(\{x, y, z, x+y+z\}) = \frac{3}{4}, \quad q_2(\{x, y, z, v, x+y+z+v\}) = \frac{5}{8}.$$
$\square$

remark. Lemma 5.8 是 Lemma 3.1 的 LD 版主工具箱，后续所有估计都依赖它。

Lemma 5.8 (Lemma 5.9 (Original)). *Let S be a set of vectors.*

1.
$$q_0(\emptyset) = 1, \qquad q_0(\{x\}) = \frac{1}{2}, \qquad q_0(S) \leq \frac{1}{4}$$
for all other S.

2. *If $m(S) = 0$ and $|S|$ is odd, then either $q_0(S) \leq 1/16$, or S is of one of the forms*
$$\{x, y, x+y\}, \quad \{x, y, z, x+y, x+z\}, \quad \{x, y, z, x+y, y+z, x+z, x+y+z\}.$$

3. *Either $J(S) = \emptyset$ or $a_0 \leq 1/4$.*

Proof. 1. If $S \neq \emptyset, \{x\}$, then $\operatorname{rank}(S) \geq 2$, so apply Lemma 5.8(1).

2. If $\operatorname{rank}(S) \geq 4$, item 1 gives $q_0(S) \leq 1/16$. If $\operatorname{rank}(S) \leq 3$, only the listed possibilities occur.

3. If $J(S) \neq \emptyset$, then $\operatorname{rank}(J(S)) \geq 2$ (there is no dependency in $\{x\}$), so item 1 implies $a_0 \leq 1/4$. □

remark. 其中前两类分别对应 *triangle* 与 K_4^-，第三类是 *LD* 场景新增的"无图论对应"异常结构。

Proof of Claim 8 (Original). The proof of Claim 2 relied on (4) and Lemmas 3.1–3.2. To prove Claim 8, replace them by **(7)** and Lemmas **5.8–5.9**. The argument then goes through line-by-line after the substitutions

$$G \mapsto S, \qquad m \mapsto m(S), \qquad H \mapsto J(S).$$

The unconditional estimates (Lemma 3.1 type) and conditional estimates (Lemma 3.2 type) both carry over.

As in Claim 2, explicit calculations are needed on exceptional small structures. Besides the direct analogues, there is one extra exceptional set:

$$S = \{x, y, z, x+y, x+z, y+z, x+y+z\},$$

for which

$$Q_S(X) = \frac{1}{8} + \frac{7}{8}X^4.$$

Hence

$$f(S) = \frac{1}{8} = \frac{1}{7} - \frac{1}{56}.$$

This completes the proof. □

remark. **难点（对应此处）**：*Claim 8* 的技术关键是"*Claim 2* 证明模板可迁移"，但必须额外处理 *7* 元异常结构这一处新分支。

> **Claim 5.9** (Claim 9 (Original)). *Let $\Lambda^{(2)}$ be the OLDC spectrum*
>
> $$\lambda_S^{(2)} = (-1)^{|S|} \left[\sum_I q_I(S) - q_\square(S) \right],$$
>
> *where I denotes a linearly independent 4-set and $\square = \{a, b, c, a+b+c\}$. Then:*
>
> 1. *$\lambda_S^{(2)} = 0$ for all $|S| \leq 3$;*
> 2. *$\lambda_S^{(2)} = 1/16$ for linearly independent $|S| = 4$;*
> 3. *$\lambda_{\{x,y,z,x+y,x+z\}}^{(2)} = 1/8$;*
> 4. *$|\lambda_S^{(2)}| \leq 1$ for all S.*

Proof. 1. Clear.

2. For any linearly independent 4-sets I, I', we have

$$q_I(I') = q_4(I') = \frac{1}{16},$$

and $q_\square(I') = 0$.

3. Let $S = \{x, y, z, x+y, x+z\}$. Then $q_I(S) = 0$ for any linearly independent 4-set I. The only subset of S isomorphic to $\square$ is

$$\{y, z, x+y, x+z\},$$

so $q_\square(S) = 1/8$.

4. $|\lambda_S^{(2)}|$ is a difference of probabilities, so it is at most 1.

$\square$

remark. Claim 9 的作用与图论中的 *Claim 3* 完全一致：抬升第一主谱上的坏 *tight* 点。

Closing The Uniform-Measure Case (Original). Using the same argument as before, if $\mathcal{F}$ is odd-LD-agreeing then

$$\mu(\mathcal{F}) \le \frac{1}{8}.$$

For odd-LD-intersecting families, Lemma 2.4 yields equality only for families containing a fixed Schur triple. For odd-LD-agreeing families, the same monotonization argument as Lemma 2.7 yields equality only for Schur juntas. Stability follows by the same argument as before.

remark. 至此，*Theorem 5.1* 的上界、等号结构与稳定性全部闭环。

5.4 Small p

Skew-Measure Extension (Original). For $p < 1/2$, Theorem 5.2 is proved by the Section 4 technique. The main-claim proof (Claim 5 type) now also uses Lemma 4.7, and its extension below.

remark. 这部分对应 *Section 4.4*：思路同构，但要补一个 *LD* 版本的有限分类引理。

Lemma 5.10 (Lemma 5.10 (Original))**.** *Let S be a set of vectors.*

1. *If $m(S) = 1$ and $|S| > 1$, then $|S| \ge 4$.*

2. *If $m(S) = 0$ and $|S| \le 5$, then S is of the form*

$$\{x, y, x+y\}, \quad \{x, y, z, x+y+z\}, \quad \{x, y, z, w, x+y+z+w\}, \quad \textit{or} \quad \{x, y, z, x+y, x+z\}.$$

Proof. 1. The smallest linear dependency is $\{x, y, x+y\}$.

2. Easy enumeration; note

$$\{x, y, z, x+y, x+y+z\}$$

is isomorphic to the last listed form.

$\square$

remark. 这四类分别对应 *triangle*、C_4、C_5、K_4^-；唯一新增要核查的是 *Lemma 5.9(2)* 中那一类 *7* 元异常结构。

Final Transfer To Theorem 5.2 (Original). All sets in Lemma 5.10(2) correspond to graph cases already controlled. For the extra exceptional case from Lemma 5.9(2), we also have

$$\lambda_G > -\frac{p^3}{1-p^3}.$$

Hence Claim 5 remains valid in the current setting, which yields Theorem 5.2.

remark. ***Section 5* 核心收获**：图论主线可完整提升到 $\mathbb{Z}_2^n$ 的奇线性依赖框架，并在均匀/偏置测度下都得到极值与稳定性。

6 Section 6 Notes

6.1 Discussion And Open Problems

Section Opening (Original). There are many intriguing generalizations of the problems discussed in this paper. The authors mention several of them and state conjectures.

remark. 这一节不是技术证明，而是把方法可能延伸到哪些方向做系统盘点。

Cross-Triangle-Intersecting Families (Original). Many intersecting-family theorems admit cross-intersecting analogues. Two families of graphs, $\mathcal{F}$ and $\mathcal{G}$, are cross-triangle-intersecting if for any

$$F \in \mathcal{F}, \quad G \in \mathcal{G},$$

the intersection $F \cap G$ contains a triangle.

remark. 这是把"同一家族内两两相交"推广为"两个家族之间两两相交"。

Conjecture 6.1 (Conjecture 1 (Original))**.** *Let $\mathcal{F}$ and $\mathcal{G}$ be cross-triangle-intersecting families of graphs on the same n vertices. Then*

$$\mu(\mathcal{F})\mu(\mathcal{G}) \le \left(\frac{1}{8}\right)^2.$$

Equality holds if and only if $\mathcal{F} = \mathcal{G}$ is a $\triangle$umvirate.

remark. 这是最自然的 *cross* 版本极值猜想，结构上完全平行主定理。

Spectral Obstacle For The Cross Case (Original). The standard Hoffman-style extension from intersecting to cross-intersecting requires one extra spectral condition:

the minimal eigenvalue $\lambda_{\min}$ must be the second largest in absolute value.

For the current tailored OCC spectrum this fails, because 3-forests have eigenvalue

$$\frac{41}{224} > \frac{1}{7}.$$

The authors suggest that using more q_R coordinates may recover the needed property, but the calculations are expected to be substantially harder.

remark. **难点（对应此处）**：*cross* 推广并非"直接套用"，卡在绝对值谱序这一条额外门槛。

The Regime $p > 1/2$ (Original). The preliminary OCC ansatz

$$\lambda_G = (-1)^{|G|} \sum_{i \geq 0} c_i q_i(G)$$

worked at $p = 1/2$ because the upper and lower bounds on $4c_3 + c_4$ imposed by 4-forests and K_4^- coincided. For $p > 1/2$, these bounds contradict each other, so a more sophisticated construction is required. The paper sees no theoretical barrier to a spectral proof for all $p \leq 3/4$, and conjectures Theorem 1.4 should hold on that whole range. It also asks why the theorem fails for $p > 3/4$ (hint: Mantel's theorem).

remark. 这一段解释了方法失效点：不是思想坏了，而是当前参数化谱不够强。

Other Intersecting Families (Original). Odd-cycle-intersecting is a special case of $\mathcal{G}$-intersecting for a graph family $\mathcal{G}$.

> **Definition 6.1** (Definition 6.1 ($m(\mathcal{G})$, Original))**.** *For a family $\mathcal{G}$ of graphs, define*
>
> $$m(\mathcal{G}) = \sup_n \{\max \mu(\mathcal{F}) : \ \mathcal{F} \text{ is a } \mathcal{G}\text{-intersecting family of graphs on } n \text{ vertices}\}.$$
>
> *For a fixed graph G, abbreviate $m(\{G\})$ to $m(G)$. Similarly, write $m_p(\mathcal{G})$ under skew product measure with parameter p.*

remark. 这个定义把各种"某结构相交"问题统一成一个函数型极值对象。

Sample Facts And Questions (Original).

- Alon observed that for every star forest G, $m(G) = 1/2$. He further conjectured existence of $\varepsilon > 0$ such that for every non-star-forest G, $m(G) < 1/2 - \varepsilon$, and noted it suffices to prove this for P_3 (the 3-edge path). The simple guess $m(P_3) = 1/8$ is false: Christofides constructed a P_3-intersecting family on six vertices with measure $17/128 > 1/8$.

- Let $\mathcal{G}_k$ be the family of non-k-colorable graphs. An obvious conjecture generalizing the main theorem is

  $$m_p(\mathcal{G}_k) = p^{\binom{k+1}{2}} \quad \text{for all } p \leq \frac{2k-1}{2k},$$

  with equality only for K_{k+1}-umvirates. The authors suggest small k cases (especially $p = 1/2$) may be approachable by their methods.

- A further conjectural direction is that $\triangle$umvirates may remain extremal even for cycle-intersecting families (not only odd-cycle-intersecting), at least when $p \leq 1/2$. At $p = 1/2$ this is already delicate: there is a neck-to-neck race with the family of all graphs having at least

  $$\frac{1}{2}\binom{n}{2} + 1$$

edges. Moreover, the non-uniform-hypergraph analogue is false: in linear-dependency-intersecting families of subsets of $\mathbb{Z}_2^n \setminus \{0\}$ one can construct measure $1/2 - o(1)$ examples, e.g. all sets of vectors with cardinality at least

$$2^{n-1} + \frac{n+1}{2}.$$

remark. 这里的关键信号是：越往一般化走，极值结构可能变、证明难度也显著上升。

A Non-Graph Structure Example (Original). For $B \subset \mathbb{Z}_n$, call a family $\mathcal{F}$ of subsets of $\mathbb{Z}_n$ B-translate-intersecting if intersections of any two sets in $\mathcal{F}$ contain a translate of B. The conjectured bound is

$$|\mathcal{F}| \leq 2^{n-|B|}.$$

It is known for interval B; Russell gave an algebraic proof. Füredi–Griggs–Holzman–Kleitman proved the case $|B| = 3$. Griggs–Walker proved that for each fixed B, the conjecture holds for infinitely many n. For most configurations B, the all-n question remains open.

remark. 这说明作者方法服务于更广的"结构性相交"范式，不局限在图模型。

Connection To Entropy? (Original). The constructed OCC spectrum can be written as

$$\sum_B c_B A_B, \qquad \sum_B c_B = 1,$$

but this is not a convex combination, since some c_B are negative. If one restricts to non-negative coefficients, the method cannot improve the bound $1/4$. The authors wonder whether this relates to the entropy bound $1/4$, given superficial similarities to the entropy-based approach in [4].

remark. ***Section 6* 核心收获**：开放问题主要集中在 *three fronts*：*cross* 版本、$p > 1/2$ 区间与更一般结构；而"负系数谱组合"可能是突破 $1/4$ 屏障的关键。